

MULTIPLICATIVE FUNCTIONS AND PROBABILISTIC NUMBER THEORY

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Abstract. The asymptotic behavior of sums of multiplicative functions on a segment of natural numbers is studied. The results obtained are applied to investigate the distribution of values of real additive functions and their fractional parts.

Basic notation

k, l, m, n, r, N, ν are natural numbers; p and q are prime numbers.

C is Euler's constant.

$f(n)$, $g(n)$ and $h(n)$ are multiplicative functions.

$$\Pi_f(x) = \prod_{p \leq x} \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r} \right).$$

$\psi(n)$ and $\phi(n)$ are real additive functions.

$\pi(x)$ is the number of primes not exceeding x .

E is a set of prime numbers.

(E) is the set of natural numbers consisting of 1 and those $n > 1$ all prime divisors of which are contained in E .

n/π_y indicates that all prime divisors of n do not exceed y .

$(n, \pi_y) = 1$ means that all prime divisors of n are greater than y .

$z(t) = z(t, \tau)$ is a solution, continuous for $t > 0$, of the differential-difference equation $tz'(t) = rz(t-1)$ with the initial condition $z(t) = 1$ for $0 \leq t \leq 1$.

$$\Pi_f^{(E)}(x) = \prod_{\substack{p \leq x \\ p \in E}} \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r} \right).$$

$n_1 = n_1(n, y)$ is the product (with respect to multiplicities) of the prime divisors of n not exceeding y ; $n_2 = n_2(n, y) = n/n_1$.

Introduction

The asymptotic behavior of mean values of complex multiplicative functions $f(n)$ plays an important role in various questions of number theory. The investigations of many authors (see [1] – [8]) are related to this problem.

For many applications, particularly in probabilistic number theory, one needs to verify for a function $f(n)$ the asymptotic formula

$$\sum_{n \leq x} f(n) = \frac{e^{-C\tau}}{\Gamma(\tau)} \frac{x}{\log x} \Pi_f(x) + o\left(\frac{x}{\log x} \Pi_{|f|}(x)\right) \quad (\text{I.1})$$

together with its various modifications and generalizations, under possibly weaker restrictions on the values of $f(n)$ at prime numbers. In this connection, the values $f(p^r)$ for $r \geq 2$ do not as a rule play a critical role; it is only necessary that they do not grow too rapidly. The constant τ appearing in the right side of (I.1) characterizes the behavior of the multiplicative function $f(n)$ at prime numbers "in the mean". For example, it can mean that

$$\sum_{p \leq x} f(p) = (\tau + o(1)) \frac{x}{\log x} \quad (\text{I.2})$$

or

$$\sum_{p \leq x} \frac{f(p) \log p}{p} = (\tau + o(1)) \log x. \quad (\text{I.3})$$

It is clear that condition (I.3) is weaker than (I.2). Under the condition (I.2) formula (I.1) was proved by the authors in [5]; this result was obtained earlier by Wirsing [1] for the case of a continuous function. Under condition (I.3) formula (I.1) is not, generally speaking, valid as the example $f(n) = n^i$ shows: (I.3) holds with $\tau = 0$, and so it would follow from (I.1) that $\sum_{n \leq x} n^i = o(x)$ (since $|n^i| = 1$), while at the same time

$$\sum_{n \leq x} n^i = \frac{x^{i+1}}{i+1} + O(\log x) = \Omega(x).$$

On the other hand, in §3 of the present work it is shown that in many cases important for applications (I.1) always does follow from (I.3).

The application of these results to probabilistic number theory (§5) permits one to significantly broaden the class of real additive functions for which one can verify necessary and sufficient conditions for the existence of a nonsingular limiting distribution related to their values at primes. This is related to the fact that for "truncations" (see [9]–[11]) one can make use of a richer set of additive functions. In particular, a new generalization of the Erdős-Kac theorem is obtained (see [12], Theorem 1).

In §§4 and 6 complex multiplicative functions of modulus not exceeding 1 are considered, and various sufficient conditions for the validity of the estimate

$$\sum_{n \leq x} f(n) = o(x) \quad (\text{I.4})$$

and its refinements are verified. In a series of cases this estimate can be proved as a consequence of formula (I.1). On the other hand, for obtaining this estimate it is often sufficient to have information about the behavior of $f(n)$ only on some set E of prime numbers. For the case where $f(p) = \beta \neq 1$ for $p \not\equiv l \pmod{D}$, where l and D are fixed relatively prime numbers, the estimate (I.4) was obtained by Corrádi and Kátai [8].

In this case, as Theorem 13 of the present work shows, the more exact assertion

$$\sum_{n \leq x} f(n) = O\left(\frac{x}{(\log x)^\alpha}\right), \quad (\text{I.5})$$

where $\alpha = \alpha(\beta) > 0$ for $\beta \neq 1$, holds.

Analogous arguments yield the following interesting fact: if from elementary considerations the existence of a set E of primes such that

$$\sum_{\substack{p \leq x \\ p \in E}} 1 \sim \delta \frac{x}{\log x}, \quad \delta > 0, \quad (\text{I.6})$$

is known, then it follows, also in an elementary (and quite uncomplicated) fashion, that

$$\sum_{n \leq x} \mu(n) = o(x),$$

which is equivalent to the prime number theorem. Thus an elementary proof of the existence of a set E satisfying condition (I.6) would quickly lead to a new elementary proof of Hadamard's theorem.

In §6, using estimate (I.4) and Weyl's criterion, we obtain theorems on the uniform distribution of fractional parts of additive functions which generalize, in this direction, well-known results of Erdős [13], Delange [2], [3], Corrádi and Kátai [8] and the authors [4], [11].

Theorems about estimates of sums of multiplicative functions of the class $\mathbb{W}$ over numbers with "small" and "large" prime divisors obtained in §§1 and 2 sharpen assertions of this type obtained earlier (see [4] and [5]) and are of independent interest.

§1. General properties of functions of class $\mathbb{W}$

An arithmetic function $f(n)$ belongs to the class $\mathbb{W}$ if it is multiplicative,

$$\sum_{p \leq x} f(p) = O\left(\frac{x}{\log x}\right), \quad (\text{1.1})$$

$$f(p^r) = O((2p)^\gamma) \text{ for } r \geq 2, \quad (\text{1.2})$$

where $\gamma < \frac{1}{2}$ is a constant.

Estimates of sums of functions of class $\mathbb{W}$ over powers of primes follows from

conditions (1.1) and (1.2).

Lemma 1. *If $f(n)$ belongs to class $\mathbb{W}$, then*

$$\sum_{\substack{p^v \leq x \\ v \geq 2}} |f(p^v)| \log p^v = O(x^{1-\alpha}), \quad \alpha = \alpha(\gamma) > 0. \quad (1.3)$$

The proof of this assertion is found in [5]. From here it is not difficult to derive a simple upper bound for the sum of $f(n)$ over a segment of natural numbers.

Theorem 1. *If $f(n)$ is a function of the class $\mathbb{W}$, then*

$$\sum_{n \leq x} f(n) \ll \frac{x}{\log x} \Pi_{|f|}(x). \quad (1.4)$$

Proof. Since

$$\begin{aligned} \sum_{n \leq x} |f(n)| \log n &= \sum_{n \leq x} |f(n)| \sum_{\substack{p^r \nmid n \\ p^r \leq n}} \log p^r = \sum_{p^r \leq x} \log p^r \sum_{\substack{n \leq x \\ n \equiv 0(p^r) \\ n \neq 0(p^{r+1})}} |f(n)| \\ &= \sum_{p^r \leq x} \log p^r |f(p^r)| \sum_{\substack{n \leq \frac{x}{p^r} \\ (n, p)=1}} |f(n)| = \sum_{n \leq x} |f(n)| \sum_{\substack{p^r \leq \frac{x}{n} \\ p \nmid n}} |f(p^r)| \log p^r, \end{aligned} \quad (1.5)$$

it follows from (1.1)–(1.3) and (1.5) that

$$\sum_{n \leq x} |f(n)| \log n \ll x \sum_{n \leq x} \frac{|f(n)|}{n} \ll x \Pi_{|f|}(x),$$

whence

$$\sum_{n \leq x} |f(n)| \ll \frac{x}{\log x} \Pi_{|f|}(x) + \int_2^x \frac{u \Pi_{|f|}(u)}{u \log^2 u} du \ll \frac{x}{\log x} \Pi_{|f|}(x).$$

In many cases formula (1.4) actually gives the precise order of growth of the sum $\sum_{n \leq x} f(n)$. In what follows we will obtain more precise estimates of this sum under more stringent restrictions on $f(n)$.

Membership of a function $f(n)$ in the class $\mathbb{W}$ allows one to investigate the asymptotic behavior of sums related to the value of this function at numbers with "small" prime divisors.

Theorem 2. *If $f(n)$ belongs to the class $\mathbb{W}$ and*

$$\frac{1}{A} (\log x)^{\frac{1}{1-2\gamma}} < y \leq x \quad (1.6)$$

(A is some constant), then

$$\sum_{\substack{n \leq x \\ n/\pi_y}} \frac{f(n)}{n} = \Pi_f(y) + O\left(\Pi_{|f|}(y) e^{-\lambda \left(\frac{\log x}{\log y}\right)}\right), \quad (1.7)$$

where $\lambda(t)$ is defined for $t \geq 1$ and

$$\lambda(t) = t \log t + t \log \log t + O(t) \text{ for } t \rightarrow \infty. \quad (1.8)$$

The proof obviously reduces to obtaining the estimate

$$\sum_{\substack{n > y^t \\ n/\pi_y}} \frac{|f(n)|}{n} \ll \Pi_{|f|}(y) e^{-\lambda(t)} \quad (1.9)$$

for

$$1 \leq t < A_1 \frac{y^{1-2\gamma}}{\log y}, \quad (1.10)$$

where A_1 is an arbitrarily large constant. If $1 \leq t \leq 3A_1$, this estimate is trivial. Let $t > 3A_1$ and $0 < \delta < 1 - 2\gamma$. Then

$$\sum_{\substack{n > y^t \\ n/\pi_y}} \frac{|f(n)|}{n} < \sum_{\substack{n > y^t \\ n/\pi_y}} \frac{|f(n)|}{n} \left(\frac{n}{y^t}\right)^\delta < y^{-t\delta} \prod_{p \leq y} \left(1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^{r(1-\delta)}}\right).$$

From this we have

$$\begin{aligned} \frac{1}{\Pi_{|f|}(y)} \sum_{\substack{n > y^t \\ n/\pi_y}} \frac{|f(n)|}{n} &< y^{-t\delta} \prod_{p \leq y} \frac{1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^{r(1-\delta)}}}{1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^r}} \\ &< y^{-t\delta} \exp \sum_{p \leq y} \sum_{r=1}^{\infty} |f(p^r)| \left(\frac{1}{p^{r(1-\delta)}} - \frac{1}{p^r} \right). \end{aligned}$$

Since

$$\sum_{p \leq x} |f(p)| \log p = O(x),$$

it follows that

$$\begin{aligned} \sum_{p \leq y} |f(p)| \left(\frac{1}{p^{1-\delta}} - \frac{1}{p} \right) &= - \sum_{p \leq y} |f(p)| \log p \int_p^y d \left(\frac{1}{u^{1-\delta} \log u} - \frac{1}{u \log u} \right) \\ &\quad + \left(\frac{1}{y^{1-\delta} \log y} - \frac{1}{y \log y} \right) \sum_{p \leq y} |f(p)| \log p \\ &= - \int_2^y \sum_{p \leq u} |f(p)| \log p d \left(\frac{1}{u^{1-\delta} \log u} - \frac{1}{u \log u} \right) + O \left(\frac{y^\delta}{\log y} \right) \\ &\ll \int_2^y u d \left(\frac{1}{u \log u} - \frac{1}{u^{1-\delta} \log u} \right) + \frac{y^\delta}{\log y} \ll \end{aligned}$$

$$\ll \int_2^y \frac{du}{u^{1-\delta} \log u} - \log \log y = \int_{2^\delta}^{y^\delta} \frac{du}{\log u} - \log \log y \ll \frac{y^\delta}{\delta \log y} + |\log(\delta \log y)|.$$

Further, taking into account that $0 < \delta < 1 - 2\gamma$, we obtain

$$\begin{aligned} \sum_{p \leq y} \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^{r(1-\delta)}} &\ll \sum_{p \leq y} \sum_{r=2}^{\infty} \frac{(2p)^{r\gamma}}{p^{r(1-\delta)}} = 4^\gamma \sum_{p \leq y} \frac{1}{p^{1-\delta-\gamma} (p^{1-\delta-\gamma} - 2^\gamma)} \\ &< 4^\gamma \sum_{p \leq y} \frac{1}{p^{1-\delta} (p^{1-2\gamma-\delta} - 1)} < \frac{4^\gamma}{1-2\gamma-\delta} \sum_{p \leq y} \frac{1}{p^{1-\delta} \log p}. \end{aligned}$$

Finally,

$$\begin{aligned} \sum_{p \leq y} \frac{1}{p^{1-\delta} \log p} &= \frac{\pi(y)}{y^{1-\delta} \log y} - \int_2^y \pi(u) d \frac{1}{u^{1-\delta} \log u} \\ &\ll - \int_2^y \text{li}(u) d \frac{1}{u^{1-\delta} \log u} \ll \int_2^y \frac{du}{u^{1-\delta} \log^2 u} = \delta \int_{2^\delta}^{y^\delta} \frac{du}{\log^2 u} \ll \frac{y^\delta}{\delta \log^2 y}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{\Pi_{|f|}(y)} \sum_{\substack{n > y^t \\ n/\pi_y}} \frac{|f(n)|}{n} &< \exp \left\{ -t\delta \log y \right. \\ &\left. + C_1 \frac{y^\delta}{\delta \log y} \left(1 + \frac{1}{(1-2\gamma-\delta) \log y} \right) + C_1 |\log(\delta \log y)| \right\}. \end{aligned} \quad (1.11)$$

It is easy to verify that if

$$3A_1 < t < A_1 \frac{y^{1-2\gamma}}{\log y},$$

then

$$0 < \frac{\log \frac{t \log t}{2A_1}}{\log y} < 1 - 2\gamma.$$

Consequently we can assume under this condition that

$$\delta = \frac{\log \frac{t \log t}{2A_1}}{\log y}.$$

Since for this

$$\frac{1}{1-2\gamma-\delta} = \frac{\log y}{\log \frac{2A_1 y^{1-2\gamma}}{t \log t}} \ll \log y,$$

it follows from (1.11) that

$$\frac{1}{\Pi_{|f|}(y)} \sum_{\substack{n \leq y^t \\ n/\pi_y}} \frac{|f(n)|}{n} < \exp \{-t \log t - t \log \log t + O(t)\},$$

and the theorem is proved.

Theorem 3. *Under the conditions of the preceding theorem, the following estimate is valid:*

$$\sum_{\substack{n \leq x \\ n/\pi_y}} f(n) \ll \frac{x}{\log y} \Pi_{|f|}(y) e^{-\lambda \left(\frac{\log x}{\log y} \right)} \quad (1.12)$$

Theorem 1 is a special case of this assertion for $y = x$.

Proof. Let $\delta < 1$. Then

$$\sum_{\substack{n \leq x \\ n/\pi_y}} |f(n)| < \sum_{\substack{n \leq x \\ n/\pi_y}} |f(n)| \left(\frac{x}{n} \right)^{1-\delta} < x^{1-\delta} \prod_{p \leq y} \left(1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^{r(1-\delta)}} \right).$$

From here, repeating the proof of the preceding theorem verbatim, we find

$$\sum_{\substack{n \leq x \\ n/\pi_y}} |f(n)| \ll x \Pi_{|f|}(y) e^{-\lambda \left(\frac{\log x}{\log y} \right)}. \quad (1.13)$$

This bound on the factor $\log y$ is weaker than necessary. On the other hand, if

$$\frac{\log x}{\log y} \geq \log \log y,$$

then, since $\lambda(t) = t \log t + t \log \log t + O(t)$, the assertion of Theorem 3 follows entirely from (1.13).

Now let $t = \log x / \log y < \log \log y$. Arguing exactly as in the proof of Theorem 1, we get

$$\begin{aligned} \sum_{\substack{n \leq y^t \\ n/\pi_y}} |f(n)| \log n &= \sum_{\substack{n \leq y^t \\ n/\pi_y}} |f(n)| \sum_{\substack{p^r \leq \frac{y^t}{n} \\ p \leq y, p \nmid n}} |f(p^r)| \log p^r \\ &\leq \sum_{\substack{n \leq y^t \\ n/\pi_y}} |f(n)| \left\{ \sum_{p \leq \min \left(\frac{y^t}{n}, y \right)} |f(p)| \log p + \sum_{\substack{p^r \leq \frac{y^t}{n} \\ r \geq 2}} |f(p^r)| \log p^r \right\}. \end{aligned}$$

Hence by Lemma 1

$$\sum_{\substack{n \leq y^t \\ n/\pi_y}} |f(n)| \log n \ll \sum_{\substack{n \leq y^t \\ n/\pi_y}} |f(n)| \left\{ \min \left(\frac{y^t}{n}, y \right) + \left(\frac{y^t}{n} \right)^{1-a} \right\} \ll$$

$$\ll \sum_{\substack{n \leq y^{t-1} \\ n/\pi_y}} |f(n)| \left\{ y + y^{-a} \frac{y^t}{n} \right\} + y^t \sum_{\substack{y^{t-1} < n \leq y^t \\ n/\pi_y}} \frac{|f(n)|}{n}.$$

According to (1.13) and the preceding theorem

$$\sum_{\substack{n \leq y^t \\ n/\pi_y}} |f(n)| \log n \ll y^t \Pi_{|f|}(y) \{e^{-\lambda(t)} + y^{-a}\} \ll y^t \Pi_{|f|}(y) e^{-\lambda(t)}, \quad (1.14)$$

since $t \leq \log \log y$. The proof is completed by applying Abel summation to (1.14).

In concluding this section we prove one more lemma needed in the future.

Lemma 2. *If $f(n)$ is a function of the class $\mathcal{W}$, then there exists a constant $A = A_f \geq 0$ such that for any x and y*

$$\Pi_{|f|}(x)/\Pi_{|f|}(y) \ll \left(\frac{\log x}{\log y} \right)^A. \quad (1.15)$$

Proof. For the sake of definiteness, let $y < x$. Then

$$\frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)} = \prod_{y < p \leq x} \left(1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^r} \right) \ll \prod_{y < p \leq x} \left(1 + \frac{|f(p)|}{p} \right) \leq \exp \sum_{y < p \leq x} \frac{|f(p)|}{p}.$$

Since

$$\begin{aligned} \sum_{y < p \leq x} \frac{|f(p)|}{p} &= - \sum_{y < p \leq x} |f(p)| \log p \int_p^x d \frac{1}{u \log u} + \frac{1}{x \log x} \sum_{y < p \leq x} |f(p)| \log p \\ &= - \int_y^x \sum_{y < p \leq u} |f(p)| \log p d \frac{1}{u \log u} + O\left(\frac{1}{\log x}\right) \\ &\ll \int_y^x \sum_{p \leq u} |f(p)| \log p \frac{du}{u^2 \log u} + O\left(\frac{1}{\log x}\right) = \log \frac{\log x}{\log y} + O\left(\frac{1}{\log x}\right), \end{aligned}$$

(1.15) is proved.

§2. Sums over numbers with "large" prime divisors

In this section we study the asymptotic behavior of the sum

$$S_f(x, y) = \sum_{\substack{n \leq x \\ (n, \pi_y) = 1}} f(n), \quad (2.1)$$

where $f(n)$ is a function of the class $\mathcal{W}$ equal in the mean to a constant at prime numbers. This problem was also considered in the works [14], [4] and [5].

In [4] the following generalization of the well-known Mangoldt function was introduced: for a multiplicative function $f(n)$, $\Lambda_f(n)$ is uniquely defined by the equation

$$f(n) \log n = \sum_{d|n} f(d) \Lambda_f\left(\frac{n}{d}\right). \quad (2.2)$$

With the aid of the function $\Lambda_f(n)$ the problem of the asymptotic behavior of sums of multiplicative functions is reduced to an integral equation. Here we apply this argument to the sum $S_f(x, y)$. As a preliminary, we prove two lemmas.

Lemma 3. *If $f(n)$ is a function of the class $\mathbb{W}$ such that*

$$\sum_{p \leq x} f(p) \log p = \tau x + O(x\rho(x)), \quad (2.3)$$

where τ is a constant, and for $x > x_0$ the functions $\rho(x)$ and $\log(1/\rho(x))/\log x$ monotonically approach zero as $x \rightarrow \infty$, then for $x_1 < y \leq x$

$$\sum_{\substack{p^r \leq x \\ p > y}} \Lambda_f(p^r) = \tau(x - y) + O(x\rho(\sqrt[4]{x})). \quad (2.4)$$

Proof. In [4] it was shown that

$$\Lambda_f(p^r) = \log p^r \sum_{m=1}^r \frac{(-1)^{m-1}}{m} \sum_{\substack{k_1 + \dots + k_m = r \\ k_i > 0}} f(p^{k_1}) \dots f(p^{k_m}). \quad (2.5)$$

In particular, $\Lambda_f(p) = f(p) \log p$. From this we have

$$\sum_{\substack{p^r \leq x \\ p > y}} \Lambda_f(p^r) = \sum_{y < p \leq x} f(p) \log p + O\left(\sum_{\substack{p^r \leq x \\ p > x_1 \\ r \geq 2}} |\Lambda_f(p^r)|\right).$$

From the identity

$$x^\alpha \rho(x) = \exp \left\{ \alpha \log \frac{1}{\rho(x)} \left(\frac{\log x}{\log \frac{1}{\rho(x)}} - \frac{1}{\alpha} \right) \right\}$$

it is clear that for any fixed $\alpha > 0$ the function $x^\alpha \rho(x)$ grows for $x > x_0(\alpha)$. Therefore from (2.3) it follows that

$$\sum_{y < p \leq x} f(p) \log p = \tau(x - y) + O(x\rho(x)) + O(y\rho(y)) = \tau(x - y) + O(x\rho(x)),$$

and the problem reduces to proving the estimate

$$\sum_{\substack{p^r \leq x \\ p > x_1, r \geq 2}} |\Lambda_f(p^r)| = O(x\rho(\sqrt[4]{x})). \quad (2.6)$$

We have

$$\sum_{\substack{p^r \leq x \\ p > x_1, r \geq 2}} |\Lambda_f(p^r)| \leq \sum_{x_1 < p \leq \sqrt{x}} \log p \sum_{2 \leq r \leq \frac{\log x}{\log p}} r \sum_{m=1}^r \frac{1}{m} \sum_{\substack{k_1 + \dots + k_m = r \\ k_i \geq 0}} |f(p^{k_1}) \dots f(p^{k_m})|.$$

Since $f \in \mathbb{W}$, we have $|f(p^k)| \leq A p^{k\gamma}$ for $k \geq 2$, where A is a constant. Hence

$$\begin{aligned} \sum_{\substack{p^r \leq x \\ p > x_1, r \geq 2}} |\Lambda_f(p^r)| &\leq \sum_{x_1 < p \leq \sqrt{x}} \log p \sum_{r \leq \frac{\log x}{\log p}} r \sum_{m=1}^r \frac{1}{m} \sum_{l=0}^m \binom{m}{l} \\ &\times \sum_{\substack{k_1 + \dots + k_m = r \\ k_1 = \dots = k_l = 1 \\ k_{l+1}, \dots, k_m \geq 2}} |f(p)|^l A^{r-l} p^{\gamma(r-l)} \leq x^\gamma \sum_{x_1 < p \leq \sqrt{x}} \log p \sum_{0 \leq l \leq \frac{\log x}{\log p}} \left| \frac{f(p)}{p^\gamma} \right|^l \\ &\times \sum_{r \leq \frac{\log x}{\log p}} r A^r \sum_{m=1}^r \frac{1}{m} \binom{m}{l} \sum_{\substack{k_1 + \dots + k_m = r \\ k_1 = \dots = k_l = 1 \\ k_{l+1}, \dots, k_m \geq 2}} 1. \end{aligned}$$

Since

$$\begin{aligned} \sum_{m=1}^r \frac{1}{m} \binom{m}{l} \sum_{\substack{k_1 + \dots + k_m = r \\ k_1 = \dots = k_l = 1 \\ k_{l+1}, \dots, k_m \geq 2}} 1 &\leq \sum_{m=1}^r \frac{1}{m} \sum_{l=0}^m \binom{m}{l} \\ &\times \sum_{\substack{k_1 + \dots + k_m = r \\ k_1 = \dots = k_l = 1 \\ k_{l+1}, \dots, k_m \geq 2}} 1 = \sum_{m=1}^r \frac{1}{m} \sum_{k_1 + \dots + k_m = r} 1 = \sum_{m=1}^r \frac{1}{m} \binom{r-1}{m-1} = \frac{2^r - 1}{r} < \frac{2^r}{r}, \end{aligned}$$

we have

$$\begin{aligned} \sum_{\substack{p^r \leq x \\ p > x_1, r \geq 2}} |\Lambda_f(p^r)| &\leq x^\gamma \sum_{x_1 < p \leq \sqrt{x}} \log p \sum_{0 \leq l \leq \frac{\log x}{\log p}} \left| \frac{f(p)}{p^\gamma} \right|^l \\ &\times \sum_{r \leq \frac{\log x}{\log p}} (2A)^r \leq x^\gamma \sum_{x_1 < p \leq \sqrt{x}} (4A)^{\frac{\log x}{\log p}} \log p \sum_{0 \leq l \leq \frac{\log x}{\log p}} \left| \frac{f(p)}{p^\gamma} \right|^l \\ &\leq x^\gamma \sum_{x_1 < p \leq \sqrt{x}} (8A)^{\frac{\log x}{\log p}} \left(1 + \left| \frac{f(p)}{p^\gamma} \right|^{\frac{\log x}{\log p}} \right) \log p \end{aligned}$$

(using the fact that $2^m > m$ for $m \geq 1$).

Thus

$$\sum_{\substack{p^r \leq x \\ p > x_1, r \geq 2}} |\Lambda_f(p^r)| \leq x^\gamma \log x \sum_{x_1 < p \leq \sqrt{x}} x^{\frac{\log 8A}{\log p}} + \sum_{x_1 < p \leq \sqrt{x}} |8Af(p)|^{\frac{\log x}{\log p}} \log p. \quad (2.7)$$

We estimate the last two sums separately. Let $\epsilon > 0$, and let x_1 be such that

$$\frac{\log 8A}{\log x_1} < \frac{1}{2} - \gamma - \epsilon.$$

Then

$$\begin{aligned} x^\gamma \log x \sum_{x_1 < p \leq \sqrt{x}} x^{\frac{\log 8A}{\log p}} &\leq x^{\gamma + \frac{\log 8A}{\log x_1}} \log x \sum_{x_1 < p \leq \sqrt{x}} 1 \\ &\ll x^{\frac{1}{2} + \gamma + \frac{\log 8A}{\log x_1}} \ll x^{1-\epsilon} \ll x \rho(\sqrt{x}), \end{aligned} \quad (2.8)$$

by virtue of the observation made at the beginning of the proof.

Further,

$$\begin{aligned} \sum_{x_1 < p \leq \sqrt{x}} |8Af(p)|^{\frac{\log x}{\log p}} \log p &= \sum_{x_1 < p \leq \sqrt{x}} + \sum_{\sqrt{x} < p \leq \sqrt{x}} \\ &= \sum_{x_1 < p \leq \sqrt{x}} |8Af(p)|^{\frac{\log x}{\log p}} \log p = x \sum_{x_1 < p \leq \sqrt{x}} \left| \frac{8Af(p)}{p} \right|^{\frac{\log x}{\log p} - 3} \left| \frac{8Af(p)}{p} \right|^3 \log p \\ &\leq x \sum_{x_1 < p \leq \sqrt{x}} \left| \frac{8Af(p)}{p} \right|^3 \log p \cdot \max_{x_1 < p \leq \sqrt{x}} \exp \left\{ \left(\frac{\log x}{\log p} - 3 \right) \log \left| \frac{8Af(p)}{p} \right| \right\}. \end{aligned}$$

From (2.3) it follows that

$$f(p) \ll \frac{p \rho(p)}{\log p},$$

and therefore x_1 can be chosen such that for $p > x_1$

$$\left| \frac{8Af(p)}{p} \right| \leq \rho(p)$$

and

$$\begin{aligned} &\max_{x_1 < p \leq \sqrt{x}} \exp \left\{ \left(\frac{\log x}{\log p} - 3 \right) \log \left| \frac{8Af(p)}{p} \right| \right\} \\ &\leq \max_{x_1 < p \leq \sqrt{x}} \exp \left\{ -\frac{\log \frac{1}{\rho(p)}}{\log p} \log x + 3 \log \frac{1}{\rho(p)} \right\} = \exp \left\{ -\log \frac{1}{\rho(\sqrt{x})} \right\} = \rho(\sqrt{x}), \end{aligned}$$

whence

$$\sum_{x_1 < p \leq \sqrt[4]{x}} |8Af(p)|^{\frac{\log x}{\log p}} \log p \ll x\rho(\sqrt[4]{x}) \sum_{p \leq \sqrt[4]{x}} \frac{|f(p)|^3}{p^3} \log p.$$

Since $f \in \mathbb{W}$, we have

$$\sum_{p \leq z} |f(p)|^3 \ll \left(\sum_{p \leq z} |f(p)| \right)^3 \ll \frac{z^3}{\log^3 z}$$

and

$$\sum_{p \leq z} \frac{|f(p)|^3}{p^3} \log p \ll \int_2^z \frac{du}{u \log^2 u} = O(1);$$

therefore

$$\sum_{x_1 < p \leq \sqrt[4]{x}} |8Af(p)|^{\frac{\log x}{\log p}} \log p \ll x\rho(\sqrt[4]{x}). \quad (2.9)$$

Finally,

$$\sum_{\sqrt[4]{x} < p \leq \sqrt{x}} |8Af(p)|^{\frac{\log x}{\log p}} \log p \ll x \log x \sum_{p > \sqrt[4]{x}} \frac{|f(p)|^2}{p^2}. \quad (2.10)$$

From (2.3) it follows that

$$\sum_{p \leq z} |f(p)|^2 \ll \frac{z\rho(z)}{\log z} \sum_{p \leq z} |f(p)| \ll \frac{z^2\rho(z)}{\log^2 z},$$

whence

$$\begin{aligned} \sum_{p > z} \frac{|f(p)|^2}{p^2} &= 2 \sum_{p > z} |f(p)|^2 \int_p^\infty \frac{du}{u^3} = 2 \int_z^\infty \sum_{z < p \leq u} |f(p)|^2 \frac{du}{u^3} \\ &\ll \int_z^\infty \frac{\rho(u)}{u \log^2 u} du \ll \frac{\rho(z)}{\log z}. \end{aligned}$$

Therefore from (2.4) we have

$$\sum_{\sqrt[4]{x} < p \leq \sqrt{x}} |8Af(p)|^{\frac{\log x}{\log p}} \log p \ll x\rho(\sqrt[4]{x}). \quad (2.11)$$

Collecting the estimates (2.1), (2.2) and (2.3), we get

$$\sum_{\substack{p' \leq x \\ p > x_1, r \geq 2}} |\Lambda_f(p')| \ll x\rho(\sqrt[4]{x}),$$

i.e. (2.6), and this implies the assertion of the lemma.

Lemma 4. Let $f(n)$ be a function of the class $\mathcal{W}$ satisfying the condition

$$\sum_{p \leq x} \frac{f(p) \log p}{p} = \tau \log x + B + O(\eta(x) \log x), \quad (2.12)$$

where B is constant, $\eta(x) > 0$, $\eta(x)$ monotonically approaches zero as $x \rightarrow \infty$. If $y = y(x) < x^\alpha$, $\alpha < 1$, $y(x) \rightarrow \infty$ and

$$\eta(y) \log \frac{\log x}{\log y} \rightarrow 0, \quad (2.13)$$

then

$$\frac{\Pi_f(x)}{\Pi_f(y)} = \left(\frac{\log x}{\log y} \right)^\tau \left\{ 1 + O\left(\eta(y) \log \frac{\log x}{\log y} + y^{2\gamma-1} \right) \right\} \quad (2.14)$$

and

$$\frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)} \geq \left(\frac{\log x}{\log y} \right)^{|\tau|} \left\{ 1 + O\left(\eta(y) \log \frac{\log x}{\log y} \right) \right\}. \quad (2.15)$$

Proof. We have

$$\begin{aligned} \frac{\Pi_f(x)}{\Pi_f(y)} &= \prod_{y < p \leq x} \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r} \right) \\ &= \exp \left\{ \sum_{y < p \leq x} \frac{f(p)}{p} + O\left(\sum_{y < p \leq x} \frac{|f(p)|^2}{p^2} \right) + O\left(\sum_{\substack{p > y \\ r \geq 2}} \frac{(2p)^{r\gamma}}{p^r} \right) \right\}. \end{aligned}$$

Since the estimate $f(p) = O(p\eta(p))$ follows from (2.12), we see that

$$\frac{\Pi_f(x)}{\Pi_f(y)} = \exp \left\{ \sum_{y < p \leq x} \frac{f(p)}{p} (1 + O(\eta(y))) + O(y^{2\gamma-1}) \right\}.$$

For the sum in the exponent we get from (2.12)

$$\begin{aligned} \sum_{y < p \leq x} \frac{f(p)}{p} &= \frac{1}{\log x} \sum_{p \leq x} \frac{f(p) \log p}{p} - \frac{1}{\log y} \sum_{p \leq y} \frac{f(p) \log p}{p} + \int_y^x \sum_{p \leq u} \frac{f(p) \log p}{p} \frac{du}{u \log^2 u} \\ &= \tau \int_y^x \frac{du}{u \log u} + O\left(\eta(y) \int_y^x \frac{du}{u \log u} \right) = (\tau + O(\eta(y))) \log \frac{\log x}{\log y}. \end{aligned}$$

This proves assertion (2.14).

Further,

$$\frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)} \geq \prod_{y < p \leq x} \left(1 + \frac{|f(p)|}{p} \right) = \exp \left\{ \sum_{y < p \leq x} \frac{|f(p)|}{p} (1 + O(\eta(y))) \right\} \geq$$

$$\begin{aligned} &\geq \exp \left\{ \left| \sum_{y < p \leq x} \frac{f(p)}{p} \right| (1 + O(\eta(y))) \right\} = \exp \left\{ (|\tau| + O(\eta(y))) \log \frac{\log x}{\log y} \right\} \\ &= \left(\frac{\log x}{\log y} \right)^{|\tau|} \left\{ 1 + O\left(\eta(y) \log \frac{\log x}{\log y}\right) \right\}. \end{aligned}$$

Remark. Comparing Lemmas 2 and 4, we see that the inequality

$$A \geq |\tau|$$

holds under the conditions of Lemma 4.

Now we can obtain asymptotic formulas for sums of functions of class $\mathcal{W}$ over numbers with "large" prime divisors. We prove here two theorems of this type, placing various restrictions on the behavior of functions at primes.

Theorem 4. Let $f(n)$ be a function of the class $\mathcal{W}$,

$$\sum_{p \leq x} f(p) \log p = \tau x + O(x\rho(x)),$$

where $\rho(x)$ satisfies the conditions of Lemma 3. Then for $t \geq 1$

$$S_f(y^t, y) = 1 + \int_1^t y^u z'(u) du + O\left(\frac{y^t \rho(\sqrt[4]{y})}{\log y} t^{A-1} \log t\right), \quad (2.16)$$

where $z(t)$ is the solution, continuous for $t > 0$, of the differential-difference equation

$$tz'(t) = \tau z(t-1)$$

with initial condition $z(t) = 1$ ($0 \leq t \leq 1$), and A is the same as in Lemma 2.

Proof. From the definition of $\Lambda_f(n)$ and Lemma 3 it follows that

$$\begin{aligned} \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} f(n) \log n &= \sum_{\substack{n \leq y^{t-1} \\ (n, \pi_y)=1}} f(n) \sum_{\substack{p^r \leq \frac{y^t}{n} \\ p > y}} \Lambda_f(p^r) \\ &= \sum_{\substack{n \leq y^{t-1} \\ (n, \pi_y)=1}} f(n) \left\{ \tau \left(\frac{y^t}{n} - y \right) + O\left(\frac{y^t}{n} \rho(\sqrt[4]{y})\right) \right\} \\ &= \tau \sum_{\substack{n \leq y^{t-1} \\ (n, \pi_y)=1}} f(n) \int_y^{\frac{y^t}{n}} du + O\left(y^t \rho(\sqrt[4]{y}) \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} \frac{|f(n)|}{n}\right) \\ &= \tau \int_y^{y^t} S_f\left(\frac{y^t}{u}, y\right) du + O\left(y^t \rho(\sqrt[4]{y}) \frac{\Pi_{|f|}(y^t)}{\Pi_{|f|}(y)}\right). \end{aligned}$$

According to Lemma 2, this implies

$$\begin{aligned} \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} f(n) \log n &= \left\{ t S_f(y^t, y) - \int_0^t S_f(y^u, y) du \right\} \log y \\ &= \tau \log y \int_1^t y^u S_f(y^{t-u}, y) du + O(y^t \rho(\sqrt[4]{y}) t^A), \end{aligned}$$

or

$$\begin{aligned} t y^{-t} S_f(y^t, y) - \int_0^t y^{-t} S_f(y^u, y) du - \tau \int_0^{t-1} y^{-u} S_f(y^u, y) du \\ = O\left(t^A \frac{\rho(\sqrt[4]{y})}{\log y}\right). \end{aligned} \quad (2.17)$$

For $t \geq 0$ we set

$$S_f(y^t, y) = 1 + \int_0^t y^u z'(u) du + y^t R(t, y). \quad (2.18)$$

Then evidently $R(t, y) = 0$ for $0 \leq t \leq 1$. Therefore from (2.17) we find (making use of the definition of $z(t)$) that for $t \geq 1$

$$t R(t, y) - \int_0^t R(u, y) y^{u-t} du - \tau \int_0^{t-1} R(u, y) du = O\left(t^A \frac{\rho(\sqrt[4]{y})}{\log y}\right), \quad (2.19)$$

whence

$$\begin{aligned} t |R(t, y)| &\leq (|\tau| + 1) \int_1^t |R(u, y)| du + O\left(t^A \frac{\rho(\sqrt[4]{y})}{\log y}\right) \\ \frac{|R(t, y)|}{t^{|\tau|+1}} - \frac{|\tau|+1}{t^{|\tau|+2}} \int_1^t |R(u, y)| du &\leq t^{A-|\tau|-2} \frac{\rho(\sqrt[4]{y})}{\log y}. \end{aligned} \quad (2.20)$$

Integrating with respect to t on the interval $[1, t]$, we find

$$\frac{1}{t^{|\tau|+1}} \int_1^t |R(u, y)| du \leq t^{A_1} \frac{\rho(\sqrt[4]{y})}{\log y},$$

where $A_1 > 0$ is a constant. This and (2.20) imply that

$$R(t, y) \leq t^{A_2} \frac{\rho(\sqrt[4]{y})}{\log y}, \quad A_2 = \text{const} > 0. \quad (2.21)$$

Further, from (2.19) we get

$$t R(t, y) - \tau \int_1^t R(u, y) du \leq \int_1^t |R(u, y)| y^{u-t} du + \int_{t-1}^t |R(u, y)| du + t^A \frac{\rho(\sqrt[4]{y})}{\log y},$$

or, setting $R(t, y) = (\rho(\sqrt[4]{y})/\log y) R_1(t, y)$,

$$tR_1(t, y) - \tau \int_1^t R_1(u, y) du \ll \int_1^t |R_1(u, y)| y^{u-t} du + \int_{t-1}^t |R_1(u, y)| du + t^A.$$

Since

$$\int_1^t |R_1(u, y)| y^{u-t} dy \ll \sup_{1 \leq u \leq t} |R_1(u, y)|,$$

it follows that

$$tR_1(t, y) - \tau \int_1^t R_1(u, y) du \ll t^A + \sup_{1 \leq u \leq t} |R_1(u, y)|. \quad (2.22)$$

From (2.21) we obtain

$$R_1(t, y) \ll t^{A_2}.$$

Let $R_1(t, y) \ll t^B$, $B > A$. Then from (2.22) we have

$$tR_1(t, y) - \tau \int_1^t R_1(u, y) du \ll t^B \quad (2.23)$$

$$\frac{R_1(t, y)}{t^\tau} - \frac{\tau}{t^{\tau+1}} \int_0^t R_1(u, y) du \ll t^{B-\operatorname{Re} \tau - 1}.$$

Since $B > A \geq |\tau| \geq \operatorname{Re} \tau$, on integrating the last inequality over the interval $[1, t]$ we find that

$$\frac{1}{t^\tau} \int_1^t R_1(u, y) du \ll t^{B-\operatorname{Re} \tau}.$$

From this and from (2.23) we get

$$R_1(t, y) \ll t^{B-1}.$$

If $B - 1 > A$, by repeating this argument we come to the estimate

$$R_1(t, y) \ll t^{B-2},$$

etc. After a finite number of steps we will have

$$R_1(t, y) \ll t^{B'}, \quad B' \leq A.$$

Now from (2.22) it follows that

$$tR_1(t, y) - \tau \int_1^t R_1(u, y) du \ll t^A, \quad (2.24)$$

whence, just as above, we find that

$$\frac{R_1(t, y)}{t^\tau} - \frac{\tau}{t^{\tau+1}} \int_1^t R_1(u, y) du \ll t^{A-\operatorname{Re} \tau-1}. \quad (2.25)$$

If $A = \operatorname{Re} \tau$, this implies

$$\frac{1}{t^\tau} \int_1^t R_1(u, y) du \ll \log t,$$

and from (2.24) we obtain

$$R_1(t, y) \ll t^{A-1} \log t.$$

If in (2.25) $A > \operatorname{Re} \tau$, then

$$\frac{1}{t^\tau} \int_1^t R_1(u, y) du \ll \frac{t^{A-\operatorname{Re} \tau} - 1}{A - \operatorname{Re} \tau} \ll t^{A-\operatorname{Re} \tau} \log t,$$

and again making use of (2.24) we find

$$R_1(t, y) \ll t^{A-1} \log t.$$

The assertion of the theorem follows from this and from (2.18).

Corollary. If $f(n)$ is a function of the class $\mathcal{W}$ and

$$\sum_{p \leq x} f(p) = (\tau + o(1)) \frac{x}{\log x},$$

then for any fixed $t \geq 1$

$$\sum_{\substack{n \leq y^t \\ n/\pi_y}} f(n) = \frac{y^t}{\log y} z'(t, \tau) \{1 + o(1)\}.$$

This assertion follows from (2.6) if one integrates the main term by parts.

Theorem 5. Let $f(n)$ be a function of the class $\mathcal{W}$ and

$$\sum_{p \leq x} \frac{f(p) \log p}{p} = \tau \log x + O(\eta(x) \log x), \quad (2.26)$$

where $\eta(x) > 0$, $\eta(x)$ monotonically approaches zero as $x \rightarrow \infty$, and $\eta(x) \log x \rightarrow \infty$.

Then for $t \geq 1$

$$\sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} \frac{f(n)}{n} = z(t) + O(t^A \eta_1(y) \log t), \quad (2.27)$$

where $z(t)$ is the same as in the preceding theorem, A is as in Lemma 2, and

$$\eta_1(y) = \frac{1}{\log y} \int_1^y \frac{\eta(u)}{u} du.$$

The proof proceeds along the same lines as for the preceding theorem. First, from

condition (2.26) we deduce the asymptotic behavior of the sum

$$\sum_{\substack{p^r \leq x \\ p > y}} \frac{\Lambda_f(p^r)}{p^r}.$$

We have

$$\sum_{\substack{p^r \leq x \\ p > y}} \frac{\Lambda_f(p^r)}{p^r} = \sum_{y < p \leq x} \frac{f(p) \log p}{p} + O \left(\sum_{\substack{p^r \leq x \\ p > x_1, r \geq 2}} \frac{|\Lambda_f(p^r)|}{p^r} \right), \quad (2.28)$$

where x_1 is a sufficiently large constant. Exactly as in the proof of Lemma 3, we find

$$\begin{aligned} \sum_{\substack{p^r \leq x \\ p > x_1, r \geq 2}} \frac{|\Lambda_f(p^r)|}{p^r} &\leq \sum_{x_1 < p \leq \sqrt{x}} \log p \sum_{2 \leq r \leq \frac{\log x}{\log p}} \frac{r}{p^r} \sum_{m=1}^r \frac{1}{m} \sum_{l=0}^m \binom{m}{l} \\ &\quad \times \sum_{\substack{k_1 + \dots + k_m = r \\ k_1 = \dots = k_l = 1 \\ k_{l+1}, \dots, k_m \geq 2}} |f(p)|^l (Ap^\gamma)^{r-l} \\ &\leq \sum_{x_1 < p \leq \sqrt{x}} \log p \sum_{0 \leq l \leq \frac{\log x}{\log p}} \left| \frac{f(p)}{p^\gamma} \right|^l \sum_{\max(2, l) \leq r \leq \frac{\log x}{\log p}} \left(\frac{2A}{p^{1-\gamma}} \right)^r \\ &\ll \sum_{x_1 < p \leq \sqrt{x}} \frac{\log p}{p^{2(1-\gamma)}} + \sum_{x_1 < p \leq \sqrt{x}} \frac{|f(p)| \log p}{p^{2-\gamma}} + \sum_{x_1 < p \leq x} \log p \sum_{2 \leq l \leq \frac{\log x}{\log p}} \left| \frac{f(p)}{p} \right|^l. \end{aligned}$$

Since the first two sums are bounded,

$$\sum_{\substack{p^r \leq x \\ p > x_1, r \geq 2}} \frac{|\Lambda_f(p^r)|}{p^r} \ll \sum_{x_1 < p \leq \sqrt{x}} \frac{|f(p)|^2 \log p}{p^2} + O(1). \quad (2.29)$$

From (2.26) it follows that $f(p) = O(p\eta(p))$, whence

$$\sum_{p \leq x} |f(p)|^2 \log p \ll x\eta(x) \sum_{p \leq x} |f(p)| \log p \ll x^2\eta(x)$$

and

$$\sum_{p \leq x} \frac{|f(p)|^2 \log p}{p^2} \ll \eta(x) + \int_1^x \frac{\eta(u)}{u} du \ll \int_1^x \frac{\eta(u)}{u} du.$$

Now from (2.28) and (2.29) we get

$$\begin{aligned} \sum_{\substack{p^r \leq x \\ p > y}} \frac{\Lambda_f(p^r)}{p^r} &= \tau \log \frac{x}{y} + O(\eta(x) \log x) + O \left(\int_1^x \frac{\eta(u)}{u} du \right) \\ &= \tau \log \frac{x}{y} + O(\eta_1(x) \log x) = \tau \log \frac{x}{y} + O(\eta_1(y) \log x). \end{aligned} \quad (2.30)$$

Further,

$$\begin{aligned}
 \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} \frac{f(n) \log n}{n} &= \sum_{\substack{n \leq y^{t-1} \\ (n, \pi_y)=1}} \frac{f(n)}{n} \sum_{\substack{p^r \leq \frac{y^t}{n} \\ p > y}} \frac{\Lambda_f(p^r)}{p^r} \\
 &= \sum_{\substack{n \leq y^{t-1} \\ (n, \pi_y)=1}} \frac{f(n)}{n} \left\{ \tau \log \frac{y^t}{n} - \tau \log y + O\left(\eta_1(y) \log \frac{y^t}{n}\right) \right\} \\
 &= \tau \sum_{\substack{n \leq y^{t-1} \\ (n, \pi_y)=1}} f(n) \int_y^{\frac{y^t}{n}} \frac{du}{u} + O\left(t \log y \cdot \eta_1(y) \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} \frac{|f(n)|}{n}\right) \\
 &= \tau \int_y^{y^t} S_f\left(\frac{y^t}{u}, y\right) \frac{du}{u} + O\left(t \eta_1(y) \log y \frac{\Pi_{|f|}(y^t)}{\Pi_{|f|}(y)}\right).
 \end{aligned}$$

By Lemma 2,

$$\begin{aligned}
 \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} \frac{f(n) \log n}{n} &= \left\{ t S_f(y^t, y) - \int_0^t S_f(y^u, y) du \right\} \log y \\
 &= \tau \log y \int_1^t S_f(y^{t-u}, y) du + O(t^{A+1} \eta_1(y) \log y),
 \end{aligned}$$

or

$$t S_f(y^t, y) - \int_0^t S_f(y^u, y) du - \tau \int_0^{t-1} S_f(y^u, y) du = O(t^{A+1} \eta_1(y)).$$

For $t \geq 0$, setting

$$S_f(y^t, y) = z(t) + r(t, y) \eta_1(y), \quad (2.31)$$

we will have

$$r(t, y) = 0 \quad (0 \leq t \leq 1);$$

and for $t \geq 1$

$$tr(t, y) - \int_0^t r(u, y) du - \tau \int_0^{t-1} r(u, y) du = O(t^{A+1}). \quad (2.32)$$

Hence, as in the proof of Theorem 4, we find

$$r(t, y) = O(t^{A_1}),$$

where $A_1 > 0$ is a constant. If $A_1 > A + 1$, then from (2.32) follows

$$tr(t, y) - (\tau + 1) \int_1^t r(u, y) du = O(t^{A_1}),$$

$$\begin{aligned} \frac{r(t, y)}{t^{\tau+1}} - \frac{\tau+1}{t^{\tau+2}} \int_1^t r(u, y) du &= O(t^{A_1 - \operatorname{Re} \tau - 2}), \\ \frac{1}{t^{\tau+1}} \int_1^t r(u, y) du &= O(t^{A_1 - \operatorname{Re} \tau - 1}), \end{aligned} \quad (2.33)$$

and from (2.33) we have

$$r(t, y) = O(t^{A_1 - 1}).$$

Reducing the degree of t in the estimate of $r(t, y)$ in this manner, in the end we arrive at the equation

$$tr(t, y) - (\tau + 1) \int_1^t r(u, y) du = O(t^{A_1 + 1}),$$

whence, as above, we get

$$r(t, y) = O(t^A \log t),$$

and from (2.31) it follows that

$$S_f(y^t, y) = z(t) + O(\eta_1(y) t^A \log t),$$

i.e. the assertion is proved.

Corollary. *If $f(n)$ is a function of the class $\mathcal{W}$ and*

$$\sum_{p \leq x} \frac{f(p) \log p}{p} = (\tau + o(1)) \log x,$$

then for any fixed $t > 0$

$$\sum_{\substack{n \leq y^t \\ (n, \pi_y) = 1}} \frac{f(n)}{n} = z(t, \tau) + o(1).$$

§3. Convolutions and asymptotic behavior of sums on segments of natural numbers

Let $h(n)$ be a function of the class $\mathcal{W}$. In order to apply the results of the first two sections to computations of the asymptotic behavior of the sum $\sum_{n \leq x} h(n)$, we make use of the obvious equality

$$\sum_{n \leq x} h(n) = \sum_{\substack{n \leq x \\ n/\pi_y}} h(n) \sum_{\substack{m \leq \frac{x}{n} \\ (m, \pi_y) = 1}} h(m).$$

If one places the same restrictions on the values of $h(p)$ as in Theorem 4, then,

after calculating the inner sum, the problem is reduced to computing the sum

$$\sum_{\substack{n \leq x \\ n/\pi_y}} \frac{h(n)}{n},$$

which approximates $\Pi_h(y)$ (the remainder is given by Theorem 2). For this, just the fact that $h(n)$ belongs to the class $\mathcal{W}$ is alone sufficient. Therefore, it is clear that this same device can be applied to sums of the form

$$\sum_{n \leq x} h_1(n_1) h_2(n_2) = \sum_{\substack{n \leq x \\ n/\pi_y}} h_1(n) \sum_{\substack{m \leq \frac{x}{n} \\ (m, \pi_y)=1}} h_2(m),$$

where $h_1(n)$ and $h_2(n)$ are functions of the class $\mathcal{W}$, and additional restrictions on the values at primes are necessary only for $h_2(n)$. This consideration is especially important for applications to probabilistic number theory (see also [5], [11]). Moreover, it is in fact sufficient to have significantly less information about the inner sum than what is contained in Theorem 4. The restrictions on $h_2(p)$ can be correspondingly weakened.

We need here one lemma which, in a certain sense, lies intermediate between Theorems 4 and 5.

Lemma 5. *Let $f(n)$ and $g(n)$ be functions of the class $\mathcal{W}$. Further, let*

$$\sum_{p \leq x} \frac{f(p) \log p}{p} = \tau_f \log x + O(\eta(x) \log x), \quad (3.1)$$

$$\sum_{p \leq x} g(p) \log p = \tau_g x + O(x\rho(x)), \quad (3.2)$$

where $\eta(x)$ and $\rho(x)$ are positive functions monotonically approaching zero, and also $\eta(x) \log x$ and $\rho(x) \log x$ grow monotonically without bound, and $(f * g)(n)$ is the arithmetic convolution of the functions f and g . Then for $t > 1$

$$\begin{aligned} \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} (f * g)(n) &= \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} f(n) - \int_0^t y^u z'_f(u) du \\ &+ \int_0^t y^u z'_{f * g}(u) du + O\left(\frac{y^t}{\log y}(\rho) \bar{V} y + \eta_1(y) t^{A_f + A_g}\right), \end{aligned} \quad (3.3)$$

where

$$z_{f * g}(t) = z(t, \tau_f + \tau_g), \quad z_f(t) = z(t, \tau_f), \quad \eta_1(y) = \int_1^y \frac{\eta(u)}{u} du.$$

Proof. Making use of the definition of the convolution and Theorem 4, we find

$$\sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} (f * g)(n) = \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} f(n) \sum_{\substack{m \leq \frac{y^t}{n} \\ (m, \pi_y)=1}} g(m) = \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} f(n) \\ + \int_0^{t-1} y^{t-u} \sum_{\substack{n \leq y^{t-1-u} \\ (n, \pi_y)=1}} \frac{f(n)}{n} z'_g \left(t - u - \frac{\log n}{\log y} \right) du + O \left(\frac{y^{t\rho(V\bar{y})}}{\log y} t^{A_f + A_g} \right).$$

Further, since

$$\int_0^{t-1} y^{t-u} \sum_{\substack{n \leq y^{t-u} \\ (n, \pi_y)=1}} \frac{f(n)}{n} z'_g \left(t - u - \frac{\log n}{\log y} \right) du = z'_g(1+0) \int_0^{t-1} y^{t-u} \sum_{\substack{n \leq y^{t-1-u} \\ (n, \pi_y)=1}} \frac{f(n)}{n} du \\ + \int_0^{t-1} \sum_{\substack{n \leq y^u \\ (n, \pi_y)=1}} \frac{f(n)}{n} du \int_0^{t-1-u} y^{t-v} z''_g(t-v-u) dv,$$

from Theorem 5 we get

$$\sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} (f * g)(n) = \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} f(n) + z'_g(1+0) \int_0^{t-1} z_f(t-u-1) y^{t-u} du \\ + \int_0^{t-1} z_f(u) du \int_0^{t-1-u} y^{t-v} z''_g(t-v-u) dv \\ + O(\eta_1(y) \int_0^{t-2} y^{t-u} (t-1-u)^{A_f} \log(t-1-u) du) \\ + O \left(\frac{y^{t\rho(V\bar{y})}}{\log y} t^{A_f + A_g} \right) + O \left(\eta_1(y) \int_1^{t-1} u^{A_f} \log u du \left| \int_0^{t-1-u} y^{t-v} z''_g(t-u-v) dv \right| \right).$$

Hence after simple computations we have

$$\sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} (f * g)(n) = \sum_{\substack{n \leq y^t \\ (n, \pi_y)=1}} f(n) + \int_0^{t-1} y^{t-u} (z'_g(1+0) z_f(t-u-1) \\ + \int_0^{t-1-u} z''_g(t-u-v) z_f(v) dv) du + O \left(\frac{y^t}{\log y} (\rho(V\bar{y}) + \eta_1(y)) t^{A_f + A_g} \right).$$

Integrating by parts the expression under the integral sign and taking into account the identity

$$z'_{f*g}(t) = z'_f(t) + z'_g(t) + \int_0^t z'_f(u) z'_g(t-u) du,$$

we obtain the proof of the lemma.

Now we can estimate the sum

$$\sum_{n \leq x} h(n_1) (f * g)(n_2),$$

where $h(n)$ is an arbitrary function of the class $\mathcal{W}$, and $f(n)$ and $g(n)$ satisfy the conditions of the preceding lemma.

Setting, as usual, $t = \log x / \log y$ and $x = y^t$, we have

$$\begin{aligned} \sum_{n \leq y^t} h(n_1) (f * g)(n_2) &= \sum_{n_1 \leq y^t} h(n_1) \sum_{\substack{n_2 \leq \frac{y^t}{n_1} \\ t - \frac{\log n_1}{\log y}}} (f * g)(n_2) \\ &= \sum_{n_1 \leq y^t} h(n_1) \sum_{\substack{n_2 \leq \frac{y^t}{n_1}}} f(n_2) - \sum_{n_1 \leq y^t} h(n_1) \int_0^{t - \frac{\log n_1}{\log y}} y^u z_f'(u) du \\ &\quad + \sum_{n_1 \leq y^t} h(n_1) \int_0^{t - \frac{\log n_1}{\log y}} y^u z_{f * g}'(u) du + O\left(\frac{y^t}{\log y} t^{A_f + A_g} (\eta_1(y) + \rho(\sqrt[4]{y})) \sum_{n_1 \leq y^t} \frac{|h(n_1)|}{n_1}\right) \\ &= \sum_{n \leq y^t} h(n_1) f(n_2) - \int_0^t y^{t-u} \sum_{\substack{n \leq y^{t-u} \\ n/\pi_y}} \frac{h(n)}{n} z_f'\left(t - u - \frac{\log n}{\log y}\right) du \\ &\quad - \int_0^t y^{t-u} \sum_{\substack{n \leq y^{t-u} \\ n/\pi_y}} \frac{h(n)}{n} z_{f * g}'\left(t - u - \frac{\log n}{\log y}\right) du \\ &\quad + O\left(\frac{y^t}{\log y} (\eta_1(y) + \rho(\sqrt[4]{y})) t^{A_f + A_g} \Pi_{|h|}(y)\right). \end{aligned}$$

Since

$$\begin{aligned} \int_0^t y^{t-u} \sum_{\substack{n \leq y^{t-u} \\ n/\pi_y}} \frac{h(n)}{n} z_f'\left(t - u - \frac{\log n}{\log y}\right) du &= \int_1^t z_f'(u) y^u \sum_{\substack{n \leq y^u \\ n/\pi_y}} \frac{h(n)}{n} du \\ &\quad - \int_1^t y^u du \int_0^{u-1} z_f''(u-v) \sum_{\substack{y^v < n \leq y^u \\ n/\pi_y}} \frac{h(n)}{n} du, \end{aligned}$$

on making use of the theorem about numbers with "small" prime divisors (Theorem 2) we get for $h \in \mathcal{W}$

$$\begin{aligned} \int_0^t y^{t-u} \sum_{\substack{n \leq y^{t-u} \\ n/\pi_y}} \frac{h(n)}{n} z_f'\left(t - u - \frac{\log n}{\log y}\right) du &= \Pi_h(y) \int_1^t y^u z_f'(u) du \\ &\quad + O\left(\Pi_{|h|}(y) \int_1^t y^u e^{-u} u^{\operatorname{Re} \tau_f - 1} du\right) + O\left(\Pi_{|h|}(y) \int_1^t y^u du \int_0^{u-1} e^{-v} (u-v)^{\operatorname{Re} \tau_f - 2} dv\right) = \end{aligned}$$

$$= \Pi_h(y) \int_1^t y^u z_f'(u) du + O\left(\frac{x}{\log x} \Pi_{|h|}(y) t^{\operatorname{Re} \tau_f - 1}\right).$$

Making use of this equation, we find

$$\begin{aligned} \sum_{n \leq y^t} h(n_1) (f * g)(n_2) &= \sum_{n \leq y^t} h(n_1) f(n_2) - \Pi_h(y) \int_1^t y^u z_f'(u) du \\ &\quad + \Pi_h(y) \int_1^t y^u z_{f * g}'(u) du + O\left(\frac{x}{\log x} \Pi_{|h|}(y) t^{\operatorname{Re} \tau_f - 1}\right) \\ &\quad + O\left(\frac{x}{\log x} \Pi_{|h|}(y) t^{\operatorname{Re}(\tau_f + \tau_g) - 1}\right) + O\left(\frac{x}{\log x} \Pi_{|h|}(y) t^{A_f + A_g - 1} (\rho(\sqrt[4]{y}) + \eta_1(y))\right). \end{aligned}$$

From this, on the basis of Lemma 4, it follows that

$$\begin{aligned} &\sum_{n \leq x} h(n_1) (f * g)(n_2) - \Pi_{|h|}(y) \int_1^t y^u z_{f * g}'(u) du \\ &= \sum_{n \leq x} h(n_1) f(n_2) - \Pi_h(y) \int_1^t y^u z_f'(u) du + O\left(\frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_f(x)}{\Pi_f(y)} \frac{1}{t}\right) \quad (3.4) \\ &\quad + O\left(\frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_f(x)}{\Pi_f(y)} \frac{\Pi_g(x)}{\Pi_g(y)} \frac{1}{t}\right) + O\left(\frac{x}{\log x} t^{A_f + A_g - 1} (\rho(\sqrt[4]{y}) + \eta_1(y))\right). \end{aligned}$$

The following assertion, fundamental for what follows, is easily obtained from (3.4).

Theorem 6. *Let $h(n)$ and $f(n)$ be multiplicative functions of the class $\mathbb{W}$, where, moreover, $f(n)$ satisfies the condition of Lemma 5. If there exists a sequence $g(p)$ such that*

$$\sum_{y < p \leq x} g(p) \log p \sim \tau_g x, \quad \sum_{p \leq x} |g(p)| = O\left(\frac{x}{\log x}\right)$$

and

$$\sum_{y < p \leq x} \frac{|f(p) - g(p)| - |f(p)|}{p} \rightarrow -\infty \quad (3.5)$$

for $y = y(x) \rightarrow \infty$ as $x \rightarrow \infty$ as quickly as desired, but such that $\log x / \log y \rightarrow \infty$, then

$$\sum_{n \leq x} h(n_1) f(n_2) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_h(y) \frac{\Pi_f(x)}{\Pi_f(y)} + o\left(\frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)}\right). \quad (3.6)$$

Proof. We extend $g(p)$ to a multiplicative function $g(n)$ by the condition $g(p^r) = 0$ for $r \geq 2$. We make use of (3.4), keeping in mind that $y^t = x$. Integrating by parts and recalling the estimates of $z_f(t)$, we get

$$\int_1^t y^u z_f'(u) du = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} t^{\tau_f} + O\left(\frac{x}{\log x} t^{\operatorname{Re} \tau_f - 1}\right). \quad (3.7)$$

On the basis of Lemma 4

$$t^{\tau_f} = \frac{\Pi_f(x)}{\Pi_f(y)} (1 + O(\eta(y) \log t)). \quad (3.8)$$

The application of Theorem 1 to the left side of (3.4) gives

$$\left| \sum_{n \leq x} h(n_1) (f * g)(n_2) - \Pi_h(y) \int_1^t y^u z'_{f * g}(u) du \right| \leq \frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_{|f * g|}(x)}{\Pi_{|f * g|}(y)} + \frac{x}{\log x} \Pi_{|h|}(y) t^{\operatorname{Re}(\tau_f + \tau_g)}. \quad (3.9)$$

Collecting the estimates (3.4), (3.7)–(3.9), we get

$$\sum_{n \leq x} h(n_1) f(n_2) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_h(y) \frac{\Pi_f(x)}{\Pi_f(y)} + O\left(\frac{x}{\log x} t^{A_f + A_g} (\rho(\sqrt[4]{y}) + \eta_1(y))\right) + O\left(\frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_{|f * g|}(x)}{\Pi_{|f * g|}(y)}\right). \quad (3.10)$$

Since $g(p^r) = 0$ and $f(p^r) = O((2p)^{r\gamma})$ for $r \geq 2$, and $f(p) = O(p/\log p)$, it follows that

$$\begin{aligned} \sum_{r=1}^{\infty} \frac{|(f * g)(p^r)|}{p^r} &= \sum_{r=1}^{\infty} \frac{|f(p^r) + g(p)f(p^{r-1})|}{p^r} = \frac{|f(p) + g(p)|}{p} \\ &+ O\left(\sum_{r=2}^{\infty} \frac{(2p)^{r\gamma}}{p^r}\right) + O\left(\frac{|g(p)|}{p \log p}\right) + O\left(|g(p)| \sum_{r=3}^{\infty} \frac{(2p)^{(r-1)\gamma}}{p^r}\right) \\ &= \frac{|f(p) + g(p)|}{p} + O\left(\frac{|g(p)|}{p \log p}\right) + O\left(\frac{1}{p^{2(1-\gamma)}}\right). \end{aligned}$$

Hence

$$\begin{aligned} \frac{\Pi_{|f * g|}(x)}{\Pi_{|f * g|}(y)} &= \exp \sum_{y < p \leq x} \log \left(1 + \sum_{r=1}^{\infty} \frac{|(f * g)(p^r)|}{p^r}\right) = \exp \left\{ \sum_{y < p \leq x} \frac{|f(p) + g(p)|}{p} \right. \\ &\quad \left. + O\left(\sum_{y < p \leq x} \frac{|f(p) + g(p)|^2}{p^2}\right) + O\left(\sum_{y < p \leq x} \frac{|g(p)|}{p \log p}\right) \right. \\ &\quad \left. + O\left(\sum_{y < p \leq x} \left(\frac{|g(p)|}{p \log p}\right)^2\right) + O(1) \right\} = \exp \left\{ \sum_{y < p \leq x} \frac{|f(p) + g(p)|}{p} + O(1) \right\}, \end{aligned} \quad (3.11)$$

since the series

$$\sum_p \frac{|g(p)|}{p \log p}, \quad \sum_p \left(\frac{|f(p) + g(p)|}{p}\right)^2$$

converge. From this and from (3.10) follows

$$\sum_{n \leq x} h(n_1) f(n_2) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_h(y) \frac{\Pi_f(x)}{\Pi_f(y)} + o\left(\frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)}\right) +$$

$$+ O\left(\frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)} \exp \sum_{y \leq p \leq x} \frac{|f(p) + g(p)| - |f(p)|}{p}\right). \quad (3.12)$$

Substituting $g(p)$ for $-g(p)$ and making use of condition (3.5), we obtain (3.6).

Theorem 6 serves as a source of many facts in the theory of mean values of multiplicative functions. We cite several examples.

Theorem 7. Let $f(n)$ be multiplicative, $h(n) \in \mathbb{V}$,

$$\sum_{p \leq x} |f(p)| = O\left(\frac{x}{\log x}\right), \quad f(p^r) = O((2p)^{r\gamma}) \quad \left(r \geq 2, \gamma < \frac{1}{2}\right)$$

$$\sum_{p \leq x} \frac{f(p) \log p}{p} \sim \tau \log x.$$

If there exists a set E such that

$$\sum_{\substack{p \leq x \\ p \in E}} 1 \sim \delta \frac{x}{\log x}, \quad \delta > 0,$$

and the values of $f(p)$ for $p \in E$ are to be found in the interior of the angle

$$\left| \arg \left(\frac{z}{z_0} - 1 \right) \right| \leq \frac{\pi}{2} - \varepsilon, \quad \varepsilon > 0,$$

with vertex at the point $z_0 \neq 0$, then

$$\sum_{n \leq x} h(n_1) f(n_2) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_h(y) \frac{\Pi_f(x)}{\Pi_f(y)} + o\left(\frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)}\right). \quad (3.13)$$

Proof. In the preceding theorem we set

$$g(p) = \begin{cases} z_0 & \text{for } p \in E, \\ 0 & \text{for } p \notin E. \end{cases}$$

If $p \in E$, then

$$\begin{aligned} |f(p)| - |f(p) - g(p)| &= |g(p)| \left(\left| \frac{f(p)}{g(p)} \right| - \left| \frac{f(p)}{g(p)} - 1 \right| \right) \\ &= |z_0| \left(\left| \frac{f(p)}{z_0} \right| - \left| \frac{f(p)}{z_0} - 1 \right| \right). \end{aligned}$$

We write $f(p)/z_0 - 1$ in exponential form: $f(p)/z_0 - 1 = re^{i\phi}$; then, according to the condition of the theorem, $|\phi| \leq \pi/2 - \epsilon$, $\epsilon > 0$, and

$$\begin{aligned} \frac{|f(p)|}{z_0} - \left| \frac{f(p)}{z_0} - 1 \right| &= |re^{i\phi} + 1| - r = \sqrt{1 + r^2 + 2r \cos \phi} - r \\ &= \frac{1 + 2r \cos \phi}{\sqrt{1 + 2r \cos \phi + r^2} + r} \geq \frac{\cos \phi + 2r \cos \phi}{\sqrt{1 + 2r + r^2} + r} = \cos \phi \geq \sin \epsilon. \end{aligned}$$

Hence

$$\sum_{y < p \leq y^t} \frac{|f(p) - g(p)| - |f(p)|}{p} = \sum_{\substack{y < p \leq y^t \\ p \in E}} \frac{|f(p) - z_0| - |f(p)|}{p} \\ \leq -|z_0| \sin \varepsilon \sum_{\substack{y < p \leq y^t \\ p \in E}} \frac{1}{p} \leq -\frac{\delta}{2} \sin \varepsilon \cdot |z_0| \log t \rightarrow -\infty,$$

if $t \rightarrow \infty$. Our assertion now follows from the preceding theorem.

Corollary. *If on a set E of positive asymptotic density the inequality $f(p) \geq c > 0$ is satisfied, then (3.13) holds.*

Actually, in this case, for $p \in E$ and $z_0 = c/2$ we have $\arg(f(p)/z_0 - 1) = 0$.

For real functions a more general fact is given by the following.

Theorem 8. *Let $f \in \mathbb{W}$, $f(p) \geq g(p)$, and*

$$\sum_{p \leq x} g(p) \log p \sim \tau_g x, \quad \sum_{p \leq x} \frac{f(p) \log p}{p} \sim \tau_f \log x.$$

Then, if $\tau_g > \tau_f - |\tau_f|$,

$$\sum_{n \leq x} f(n) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_f(x) + o\left(\frac{x}{\log x} \Pi_{|f|}(x)\right).$$

Actually, in this case

$$\sum_{y < p \leq x} \frac{|f - g| - |f|}{p} = \sum_{y < p \leq x} \frac{f - g - |f|}{p} \\ \leq \frac{1}{2} (\tau_f - \tau_g - |\tau_f|) \log \frac{\log x}{\log y} \rightarrow -\infty \text{ for } x \rightarrow \infty.$$

Remark. The following assertion is a natural extension of Theorem 8.

Let $f \in \mathbb{W}$, $f(p) \leq g(p)$, and

$$\sum_{p \leq x} g(p) \log p \sim \tau_g x, \quad \sum_{p \leq x} \frac{f(p) \log p}{p} \sim \tau_f \log x.$$

Then, if $\tau_g < \tau_f + |\tau_f|$,

$$\sum_{n \leq x} f(n) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_f(x) + o\left(\frac{x}{\log x} \Pi_{|f|}(x)\right).$$

In connection with this result we state the following conjecture:

If $f \in \mathbb{W}$, $f(p) \leq g(p)$, and

$$\sum_{p \leq x} g(p) \log p \sim \tau_g x, \quad \sum_{p \leq x} \frac{f(p) \log p}{p} \sim \tau_f \log x, \quad \tau_f \neq 0,$$

then

$$\sum_{n \leq x} f(n) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_f(x) + o\left(\frac{x}{\log x} \Pi_{|f|}(x)\right).$$

Theorem 9. If $h(n) \in W$, $f(p^r) = O((2p)^{r^\gamma})$ ($r \geq 2$),

$$\sum_{p \leq x} \frac{f(p) \log p}{p} \sim \tau_f \log x \quad (3.14)$$

and

$$\overline{\lim}_{p \rightarrow \infty} |f(p)| - \underline{\lim}_{p \rightarrow \infty} |f(p)| < 2 |\tau_f|, \quad (3.15)$$

then there exists $y = y(x) < x$ such that $\log y = o(\log x)$ and

$$\sum_{n \leq x} h(n_1) f(n_2) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)} + o\left(\frac{x}{\log x} \Pi_{|h|}(y) \frac{\Pi_{|f|}(x)}{\Pi_{|f|}(y)}\right).$$

For the proof we set $\overline{\lim} |f(p)| = L$ and $\underline{\lim} |f(p)| = l$. We have

$$\begin{aligned} \sum_{y < p \leq y^t} \frac{|f(p)|^2}{p} \sum_{y < p \leq y^t} \frac{1}{p} - \left(\sum_{y < p \leq y^t} \frac{|f(p)|}{p} \right)^2 &= \frac{1}{2} \sum_{\substack{y < p \leq y^t \\ -y < q \leq y^t}} \frac{(|f(p)| - |f(q)|)^2}{pq} \\ &\leq \max_E (L - l + 2\varepsilon)^2 \sum_{\substack{y < p \leq y^t \\ p \in E}} \frac{1}{p} \cdot \left(\sum_{y < p \leq y^t} \frac{1}{p} - \sum_{\substack{y < p \leq y^t \\ p \in E}} \frac{1}{p} \right) \\ &\leq \frac{1}{4} (L - l + 2\varepsilon)^2 \left(\sum_{y < p \leq y^t} \frac{1}{p} \right)^2 \leq \frac{1}{4} (L - l + 2\varepsilon) (1 + \varepsilon)^2 \log^2 t, \end{aligned}$$

since the positive definite quadratic form

$$\sum_{\substack{y < p \leq y^t \\ y < q \leq y^t}} \frac{(x_p - x_q)^2}{pq}$$

in the cube $l - \varepsilon \leq x_p \leq L + \varepsilon$ reaches its maximum at one of the vertices. Hence on the basis of (3.15) we get

$$\begin{aligned} \sum_{y < p \leq y^t} \frac{|f(p)|^2}{p} \sum_{y < p \leq y^t} \frac{1}{p} - \left(\sum_{y < p \leq y^t} \frac{|f(p)|}{p} \right)^2 \\ \leq (|\tau_f|^2 - \varepsilon_2) \log^2 t = |\tau_f|^2 \left(\sum_{y < p \leq y^t} \frac{1}{p} \right)^2 - \varepsilon_1 \log^2 t \end{aligned}$$

for some $\varepsilon_1 > 0$, or

$$\sum_{y < p \leq y^t} \frac{|f(p)|^2 - |\tau_f|^2}{p} \sum_{y < p \leq y^t} \frac{1}{p} - \left(\sum_{y < p \leq y^t} \frac{|f(p)|}{p} \right)^2 \leq -\varepsilon_1 \log^2 t.$$

By virtue of the Cauchy-Bunjakovskii inequality and (3.14),

$$\begin{aligned}
& \left(\sum_{y < p \leq y^t} \frac{|f(p) - \tau_f|}{p} \right)^2 - \left(\sum_{y < p \leq y^t} \frac{|f(p)|}{p} \right)^2 \\
& \leq \sum_{y < p \leq y^t} \frac{|f(p) - \tau_f|^2}{p} \sum_{y < p \leq y^t} \frac{1}{p} - \left(\sum_{y < p \leq y^t} \frac{|f(p)|}{p} \right)^2 \\
& = \sum_{y < p \leq y^t} \frac{|f(p)|^2 - |\tau_f|^2}{p} \sum_{y < p \leq y^t} \frac{1}{p} + \varepsilon_3 \log^2 t - \left(\sum_{y < p \leq y^t} \frac{|f(p)|}{p} \right)^2 \\
& \leq -\varepsilon_4 \log^2 t, \quad \varepsilon_4 > 0.
\end{aligned}$$

Dividing both parts by

$$\sum_{y < p \leq y^t} \frac{|f(p) - \tau_f|}{p} + \sum_{y < p \leq y^t} \frac{|f(p)|}{p}$$

and calculating that this expression is $O(\log t)$, we get the required assertion.

Corollary 1. *If $f(n)$ is multiplicative,*

$$|f(p)| = \text{const}, \quad f(p^r) = O((2p)^{\gamma})$$

and

$$\sum_{p \leq x} \frac{f(p) \log p}{p} \sim \tau_f \log x, \quad \tau_f \neq 0,$$

then

$$\sum_{p \leq x} f(n) = \frac{e^{-C\tau_f}}{\Gamma(\tau_f)} \frac{x}{\log x} \Pi_f(x) + o\left(\frac{x}{\log x} \Pi_{|f|}(x)\right).$$

Corollary 2. *In the preceding corollary the condition that $|f(p)|$ is constant can be replaced by the condition*

$$\overline{\lim}_{p \rightarrow \infty} |f(p)| < 2|\tau_f|.$$

In particular, a well-known theorem of Delange ([2], [6]) and its generalization are contained in the following: if $|f(p)| \leq \tau$ and the series $\sum_p (f(p) - \tau)/p$ converges, then

$$\sum_{n \leq x} f(n) \sim Cx (\log x)^{\tau-1}, \quad C = \text{const}.$$

In Theorems 6–9 in addition to the “natural” requirements on $f(n)$:

$$\begin{aligned}
& \sum_{p \leq x} \frac{f(p) \log p}{p} \sim \tau \log x, \quad \sum_{p \leq x} |f(p)| = O\left(\frac{x}{\log x}\right), \\
& f(p^r) = O((2p)^{\gamma}) \quad \left(r \geq 2, \gamma < \frac{1}{2}\right),
\end{aligned}$$

there are additional restrictions on the set of values $f(p)$. These restrictions are not, as it might seem at first glance, exclusively related to the method of proof. We consider the following example. Let E be the set of prime numbers included in the intervals

$(e^{n^2}, e^{n^2+1}), n = 1, 2, 3, \dots,$

$$f(p) = \begin{cases} 1, & \text{if } p \in E, \\ 0, & \text{if } p \notin E, \end{cases}$$

where $f(p^r) = 0$ for $r \geq 2$. Then the series $\sum_p f(p)/p$ converges. Actually

$$\sum_{e^{n^2} < p < e^{n^2+1}} \frac{1}{p} = \log \frac{\log e^{n^2+1}}{\log e^{n^2}} + O\left(\frac{1}{\log e^{n^2}}\right) = O\left(\frac{1}{n^2}\right).$$

It follows that $\Pi_f(x) = O(1)$ and

$$\sum_{p \leq x} \frac{f(p) \log p}{p} = o(\log x),$$

i.e. condition (3.14) is satisfied with $\tau = 0$. If formula (3.13) were valid, it would imply

$$\sum_{p \leq x} f(p) < \sum_{n \leq x} f(n) = o\left(\frac{x}{\log x}\right).$$

At the same time, for $x = e^{n^2+1}$ we have

$$\begin{aligned} \sum_{p \leq x} f(p) &\geq \sum_{e^{n^2} < p < e^{n^2+1}} 1 = \pi(e^{n^2+1}) - \pi(e^{n^2}) \\ &= \frac{e^{n^2+1}}{n^2+1} - \frac{e^{n^2}}{n^2} + O\left(\frac{e^{n^2}}{n^4}\right) = \frac{e^{n^2+1}}{n^2+1} \left(1 - \frac{1}{e}\right) + O\left(\frac{e^{n^2+1}}{(n^2+1)^3}\right) > \frac{1}{2} \frac{x}{\log x}. \end{aligned}$$

§4. Functions of modulus not exceeding one

The results of the preceding section can be formulated somewhat more simply for those functions important in probabilistic number theory. For example, we have

Theorem 10. *If $|f(n)| \leq 1$, $|f(p)| = 1$ for all p and*

$$\sum_{p \leq x} \frac{f(p) \log p}{p} \sim \tau \log x, \quad \tau \neq 0, \quad (4.1)$$

then

$$\sum_{n \leq x} f(n) = \frac{e^{-c\tau}}{\Gamma(\tau)} \frac{x}{\log x} \Pi_f(x) + o(x). \quad (4.2)$$

This follows immediately from Corollary 1 of Theorem 9.

In an analogous manner we get

Theorem 11. *If $h(n)$ is multiplicative, $|h(n)| \leq 1$, and $f(n)$ satisfies the conditions of the preceding theorem, then there exists $y = y(x) \rightarrow \infty$ such that $\ln y = o(\ln x)$ and*

$$\sum_{n \leq x} h(n_1) f(n_2) = \frac{e^{-C\tau}}{\Gamma(\tau)} \frac{x}{\log x} \Pi_h(y) \frac{\Pi_f(x)}{\Pi_f(y)} + o(x). \quad (4.3)$$

It is well known that the prime number theorem is in a definite sense equivalent to the bound

$$\sum_{n \leq x} \mu(n) = o(x). \quad (4.4)$$

If from some elementary considerations or other one could establish the existence of at least one set of prime numbers E satisfying the condition

$$\sum_{\substack{p \leq x \\ p \in E}} 1 \sim \delta \frac{x}{\log x}, \quad \delta > 0, \quad (4.5)$$

then the application of Theorem 7 to the functions $h(n) = f(n) = \mu(n)$ and $z_0 = -\frac{1}{2}$ would give a new elementary proof of the prime number theorem.

In what follows we will be interested in conditions for which the bound

$$\sum_{n \leq x} f(n) = o(x) \quad (4.6)$$

holds, where $f(n)$ is a multiplicative function (see also [3], [7], [8] or [11]).

Certain results of that type can be deduced from Theorems 10 and 11. On the other hand, to get (4.12) it is sufficient to have corresponding information about the values of the function only on some set of primes having positive asymptotic density.

Theorem 12. *Let $f(n)$ be multiplicative, $|f(n)| \leq 1$, E a set of primes such that*

$$\sum_{\substack{p \leq x \\ p \in E}} \frac{\log p}{p} \sim \delta \log x, \quad \delta > 0, \quad (4.7)$$

and

$$\sum_{\substack{p \leq x \\ p \in E}} f(p) = (\tau + o(1)) \frac{x}{\log x}. \quad (4.8)$$

Then, if

$$\sum_{p \in E} \frac{1 - \operatorname{Re} f(p)}{p} = \infty, \quad (4.9)$$

it follows that

$$\sum_{n \leq x} f(n) = o(x).$$

Proof. Let (E) be the set consisting of 1 and those natural numbers all prime divisors of which belong to E . Then from (4.8) it follows that

$$\sum_{\substack{n \leq x \\ n \in (E)}} f(n) = \frac{e^{-C\tau}}{\Gamma(\tau)} \frac{x}{\log x} \Pi_f^{(E)}(x) + o\left(\frac{x}{\log x} \Pi_{|f|}^{(E)}(x)\right), \quad (4.10)$$

where

$$\Pi_f^{(E)}(x) = \prod_{\substack{p \leq x \\ p \in E}} \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r}\right).$$

From (4.1) it follows that

$$\begin{aligned} \Pi_f^{(E)}(x) &\ll \prod_{\substack{p \leq x \\ p \in E}} \left(1 + \frac{f(p)}{p}\right) \ll \Pi_1^{(E)}(x) \prod_{\substack{p \leq x \\ p \in E}} \left(1 + \frac{f(p) - 1}{p}\right) \\ &\ll \Pi_1^{(E)}(x) \exp \sum_{\substack{p \leq x \\ p \in E}} \frac{\operatorname{Re} f(p) - 1}{p} = o(\Pi_1^{(E)}(x)). \end{aligned}$$

Since clearly

$$\Pi_{|f|}^{(E)}(x) \leq \Pi_1^{(E)}(x),$$

from (4.10) it follows that

$$\sum_{\substack{n \leq x \\ n \in (E)}} f(n) = o\left(\frac{x}{\log x} \Pi_1^{(E)}(x)\right). \quad (4.11)$$

We note that since

$$\sum_{\substack{n \leq x \\ n \in (E)}} 1 \sim \frac{e^{-C\delta}}{\Gamma(\delta)} \frac{x}{\log x} \Pi_1^{(E)}(x),$$

(4.11) means that

$$\sum_{\substack{n \leq x \\ n \in (E)}} f(n) = o\left(\sum_{\substack{n \leq x \\ n \in (E)}} 1\right). \quad (4.12)$$

From here it is not difficult to prove the desired assertion by applying "summation under the hyperbola".

If $yz = x$, then

$$\sum_{n \leq x} f(n) = \sum_{\substack{n \leq x \\ n \in (E)}} f(n) \sum_{\substack{m \leq \frac{x}{n} \\ m \in (E)}} f(m) = \sum_{\substack{m \leq y \\ n \in (E)}} f(n) \sum_{\substack{m \leq \frac{x}{n} \\ m \in (E)}} f(m) +$$

$$+ \sum_{\substack{m \leq z \\ m \in (E)}} f(m) \sum_{\substack{n \leq \frac{x}{m} \\ n \in (\bar{E})}} f(n) - \sum_{\substack{n \leq y \\ n \in (\bar{E})}} f(n) \sum_{\substack{m \leq z \\ m \in (E)}} f(m). \quad (4.13)$$

From (4.13) it follows that

$$\sum_{\substack{m \leq x \\ m \in (E)}} f(m) = O\left(\varepsilon(x) \frac{x}{\log x} \Pi_1^{(E)}(x)\right),$$

where $\varepsilon(x)$ is a positive function converging monotonically to zero. Therefore

$$\begin{aligned} \sum_{\substack{n \leq y \\ n \in (\bar{E})}} f(n) \sum_{\substack{m \leq \frac{x}{n} \\ m \in (E)}} f(m) &\ll x \varepsilon\left(\frac{x}{y}\right) \sum_{\substack{n \leq y \\ n \in (\bar{E})}} \frac{1}{n \log \frac{x}{n}} \Pi_1^{(E)}\left(\frac{x}{n}\right) \\ &\ll \frac{x \varepsilon\left(\frac{x}{y}\right) \Pi_1^{(E)}(x)}{(\log x)^\delta} \sum_{\substack{n \leq y \\ n \in (\bar{E})}} \frac{1}{n} \left(\log \frac{x}{n}\right)^{\delta-1} \\ &\ll \frac{x \varepsilon\left(\frac{x}{y}\right) \Pi_1^{(E)}(x) \left(\log \frac{x}{y}\right)^{\delta-1}}{(\log x)^\delta} \sum_{\substack{n \leq y \\ n \in (\bar{E})}} \frac{1}{n} \ll \frac{x \varepsilon(z) \Pi_1^{(E)}(x) (\log z)^{\delta-1}}{(\log x)^\delta} \Pi_1^{(E)}(y) \\ &\ll x \varepsilon(z) \Pi_1^{(E)}(y) \Pi_1^{(\bar{E})}(y) \frac{(\log z)^{\delta-1}}{(\log x)^\delta} \left(\frac{\log x}{\log y}\right)^\delta \ll x \varepsilon(z) \left(\frac{\log z}{\log y}\right)^{\delta-1} \end{aligned} \quad (4.14)$$

by virtue of Lemma 4 and the fact that

$$\Pi_1^{(E)}(y) \Pi_1^{(\bar{E})}(y) = \Pi_1(y) = O(\log y).$$

Further, using Theorem 1 and Lemma 4, we get

$$\begin{aligned} \sum_{\substack{m \leq z \\ m \in (E)}} f(m) \sum_{\substack{n \leq \frac{x}{m} \\ n \in (\bar{E})}} f(n) &\ll \sum_{\substack{m \leq z \\ m \in (E)}} \sum_{\substack{n \leq \frac{x}{m} \\ n \in (\bar{E})}} 1 \\ &\ll x \sum_{\substack{m \leq z \\ m \in (E)}} \frac{1}{m \log \frac{x}{m}} \Pi_1^{(\bar{E})}\left(\frac{x}{m}\right) \ll x \Pi_1^{(\bar{E})}\left(\frac{x}{z}\right) \sum_{\substack{m \leq z \\ m \in (E)}} \frac{1}{m \log \frac{x}{m}} \left(\frac{\log \frac{x}{m}}{\log \frac{x}{z}}\right)^{1-\delta} \\ &\ll \frac{x \Pi_1^{(\bar{E})}(y)}{\log y} \sum_{\substack{m \leq z \\ m \in (E)}} \frac{1}{m} \ll \frac{x \Pi_1^{(\bar{E})}(y) \Pi_1^{(\bar{E})}(z)}{\log y} \ll x \left(\frac{\log z}{\log y}\right)^\delta. \end{aligned} \quad (4.15)$$

From (4.13)–(4.15) it follows that

$$\sum_{n \leq x} f(n) \ll x \left\{ \varepsilon(z) \left(\frac{\log y}{\log z} \right)^{1-\delta} + \left(\frac{\log z}{\log y} \right)^{\delta} \right\}.$$

Choosing y and z according to the condition $\varepsilon(z) = \log z / \log y$, we get

$$\sum_{n \leq x} f(n) = o(x).$$

Corollary. If $f(n)$ satisfies condition (4.8), where $\tau \neq \delta$, then

$$\sum_{n \leq x} f(n) = o(x).$$

Actually, for this case $\operatorname{Re} \tau < \delta$ and

$$\sum_{\substack{p \leq x \\ p \in E}} \frac{1 - \operatorname{Re} f(p)}{p} \sim (\delta - \operatorname{Re} \tau) \log \log x \rightarrow +\infty.$$

For the particular case where E is the set of primes from some primitive progression and $f(p) = \beta \neq 1$ for $p \in E$, the bound $\sum_{n \leq x} f(n) = o(x)$ was proved by other methods by Kátai and Corrádi [8]. As the following theorem shows, a more precise estimate is valid in this case.

Theorem 13. Let $f(n)$ be a multiplicative function, $|f(n)| \leq 1$ and $f(p) = \beta$ for $p \equiv l \pmod{k}$, $(k, l) = 1$. Then

$$\sum_{n \leq x} f(n) \ll \begin{cases} x (\log x)^{\frac{\operatorname{Re} \beta - 1}{\varphi(k)}}, & \text{if } \operatorname{Re} \beta > 0, \\ x (\log x)^{-\frac{1}{\varphi(k)}} \log \log x, & \text{if } \operatorname{Re} \beta = 0, \\ x (\log x)^{-\frac{1}{\varphi(k)}}, & \text{if } \operatorname{Re} \beta < 0. \end{cases} \quad (4.16)$$

Proof. Evidently

$$\sum_{n \leq x} f(n) = \sum_{\substack{n \leq x \\ p/n \rightarrow p \neq l(k)}} f(n) \sum_{\substack{m \leq \frac{x}{n} \\ p/m \rightarrow p \equiv l(k)}} f(m) = \sum_{n \leq \frac{x}{2}} + \sum_{\frac{x}{2} < n \leq x}.$$

We have

$$\sum_{\frac{x}{2} < n \leq x} \ll \sum_{\substack{\frac{x}{2} < n \leq x \\ p/n \rightarrow p \neq l(k)}} \sum_{m \leq \frac{x}{n}} 1 = \sum_{\substack{\frac{x}{2} < n \leq x \\ p/n \rightarrow p \equiv l(k)}} 1 \ll$$

$$\leq \sum_{\substack{n \leq x \\ p/n \rightarrow p \neq l(k)}} 1 \ll x (\log x)^{-\frac{1}{\varphi(k)}}, \quad \sum_{n \leq \frac{x}{2}} = \sum_{n \leq \sqrt{x}} + \sum_{\sqrt{x} < n \leq \frac{x}{2}}. \quad (4.17)$$

Since

$$\sum_{\substack{m \leq y \\ p/m \rightarrow p \equiv l(k)}} f(m) = C_1 y (\log y)^{\frac{\text{Re } \beta}{\varphi(k)} - 1} + O(y (\log y)^{\frac{\text{Re } \beta}{\varphi(k)} - 2}) \quad (4.18)$$

(an obvious corollary of Theorem 2.1.4 in [4]), we have

$$\begin{aligned} \sum_{\substack{n \leq \sqrt{x} \\ p/n \rightarrow p \neq l(k)}} f(n) & \sum_{\substack{m \leq \frac{x}{n} \\ p/m \rightarrow p \equiv l(k)}} f(m) \ll x \sum_{\substack{n \leq \sqrt{x} \\ p/n \rightarrow p \neq l(k)}} \frac{1}{n} \left(\log \frac{x}{n} \right)^{\frac{\text{Re } \beta}{\varphi(k)} - 1} \\ & \ll x (\log x)^{\frac{\text{Re } \beta}{\varphi(k)} - 1} \sum_{\substack{n \leq \sqrt{x} \\ p/n \rightarrow p \neq l(k)}} \frac{1}{n} \ll x (\log x)^{\frac{\text{Re } \beta - 1}{\varphi(k)}} \end{aligned} \quad (4.19)$$

Further,

$$\begin{aligned} \sum_{\substack{\sqrt{x} < n \leq \frac{x}{2} \\ p/n \rightarrow p \neq l(k)}} f(n) & \sum_{\substack{m \leq \frac{x}{n} \\ p/m \rightarrow p \equiv l(k)}} f(m) \ll x \sum_{\substack{\sqrt{x} < n \leq \frac{x}{2} \\ p/n \rightarrow p \neq l(k)}} \frac{1}{n} \left(\log \frac{x}{n} \right)^{\frac{\text{Re } \beta}{\varphi(k)} - 1} \\ & = x (\log \sqrt{x})^{\frac{\text{Re } \beta}{\varphi(k)} - 1} \sum_{\substack{\sqrt{x} < n \leq \frac{x}{2} \\ p/n \rightarrow p \neq l(k)}} \frac{1}{n} - x \sum_{\substack{\sqrt{x} < n \leq \frac{x}{2} \\ p/n \rightarrow p \equiv l(k)}} \frac{1}{n} \int_{x/n}^{\sqrt{x}} d(\log u)^{\frac{\text{Re } \beta}{\varphi(k)} - 1} \\ & \ll -x \int_{\frac{x}{u} < n \leq \frac{x}{2}}^{\sqrt{x}} \sum_{\substack{p/n \rightarrow p \neq l(k)}} \frac{1}{n} d(\log u)^{\frac{\text{Re } \beta}{\varphi(k)} - 1} + x (\log x)^{\frac{\text{Re } \beta - 1}{\varphi(k)}}. \end{aligned}$$

Since for $2 \leq u \leq \sqrt{x}$

$$\begin{aligned} \sum_{\substack{\frac{x}{u} < n \leq \frac{x}{2} \\ p/n \rightarrow p \neq l(k)}} \frac{1}{n} & = C_2 \left(\log \frac{x}{2} \right)^{1 - \frac{1}{\varphi(k)}} - C_2 \left(\log \frac{x}{u} \right)^{1 - \frac{1}{\varphi(k)}} + O((\log x)^{-\frac{1}{\varphi(k)}}) \\ & = C_2 (\log x)^{1 - \frac{1}{\varphi(k)}} \left\{ \left(1 - \frac{\log 2}{\log x} \right)^{1 - \frac{1}{\varphi(k)}} - \left(1 - \frac{\log u}{\log x} \right)^{1 - \frac{1}{\varphi(k)}} \right\} + \end{aligned}$$

$$+ O((\log x)^{-\frac{1}{\varphi(k)}}) \ll \log u (\log x)^{-\frac{1}{\varphi(k)}},$$

we see that

$$\sum_{\substack{V\bar{x} < n \leq \frac{x}{2} \\ p/n \rightarrow p \equiv l(k)}} f(n) \sum_{\substack{m \leq \frac{x}{n} \\ p/m \rightarrow p \equiv l(k)}} f(m) \ll x (\log x)^{-\frac{1}{\varphi(k)}} \int_2^{\sqrt{x}} (\log u)^{\frac{\operatorname{Re} \beta}{\varphi(k)} - 1} \frac{du}{u}$$

$$\ll \begin{cases} x (\log x)^{\frac{\operatorname{Re} \beta - 1}{\varphi(k)}}, & \text{if } \operatorname{Re} \beta > 0, \\ x (\log x)^{-\frac{1}{\varphi(k)}} \log \log x, & \text{if } \operatorname{Re} \beta = 0, \\ x (\log x)^{-\frac{1}{\varphi(k)}}, & \text{if } \operatorname{Re} \beta < 0. \end{cases} \quad (4.20)$$

Combining the estimates (4.17), (4.19) and (4.20), we get the assertion being proved.

In concluding this section, we deduce a sharpening of the following theorem of Kátai and Corrádi [7]: if $f(n)$ is a multiplicative function of modulus not exceeding 1,

$$\sum_{p \leq x} \frac{|f(p) - \tau|}{p} = o(\log \log x), \quad (4.21)$$

where $\tau \neq 1$, then

$$\sum_{n \leq x} f(n) = o(x).$$

Theorem 14. *Under the preceding hypotheses the estimate*

$$\sum_{n \leq x} f(n) = O\left(x (\log x)^{-\frac{1 - \operatorname{Re} \tau}{2 - \operatorname{Re} \tau} + \varepsilon}\right) \quad (4.22)$$

is valid.

Proof. Let

$$h(n) = \sum_{d|n} \mu(d) \tau^{\Omega(d)} f\left(\frac{n}{d}\right),$$

i.e.

$$f(n) = \sum_{d|n} \tau^{\Omega(d)} h\left(\frac{n}{d}\right).$$

Since $|\tau| \leq 1$, we have $\mu(n) \tau^{\Omega(n)} \in \mathbb{W}$ and consequently $h(n) \in \mathbb{W}$. According to Theorem 1,

$$\begin{aligned} \sum_{n \leq x} |h(n)| &\ll \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{|h(p)|}{p}\right) = \frac{x}{\log x} \prod_{p \leq x} \left(1 + \frac{|f(p) - \tau|}{p}\right) \\ &\ll \frac{x}{\log x} \exp \sum_{p \leq x} \frac{|f(p) - \tau|}{p} \ll x (\log x)^{\varepsilon-1}. \end{aligned} \quad (4.23)$$

Further,

$$\begin{aligned} \sum_{n \leq x} f(n) &= \sum_{n \leq x} h(n) \sum_{m \leq \frac{x}{n}} \tau^{\Omega(m)} = \sum_{n \leq y} h(n) \sum_{m \leq \frac{x}{n}} \tau^{\Omega(m)} \\ &\quad + \sum_{m \leq \frac{x}{y}} \tau^{\Omega(m)} \sum_{n \leq \frac{x}{m}} h(n) - \sum_{n \leq y} h(n) \sum_{m \leq \frac{x}{y}} \tau^{\Omega(m)}. \end{aligned}$$

Using the well-known bound

$$\sum_{m \leq x} \tau^{\Omega(m)} \ll x (\log x)^{\operatorname{Re} \tau - 1}$$

and (4.23), we find

$$\begin{aligned} \sum_{n \leq x} f(n) &\ll x \sum_{n \leq y} \frac{|h(n)|}{n} \left(\log \frac{x}{n}\right)^{\operatorname{Re} \tau - 1} + x \sum_{m \leq \frac{x}{y}} \frac{1}{m} \left(\log \frac{x}{m}\right)^{\varepsilon-1} \\ &\quad + y (\log y)^{\varepsilon-1} \frac{x}{y} \left(\log \frac{x}{y}\right)^{\operatorname{Re} \tau - 1} \ll x \left(\log \frac{x}{y}\right)^{\operatorname{Re} \tau - 1} (\log y)^{\varepsilon} + x (\log y)^{\varepsilon-1} \log \frac{x}{y}. \end{aligned}$$

Choosing $y = x \cdot \exp \{- (\log x)^{1/(2 - \operatorname{Re} \tau)}\}$, we get (4.22).

§5. Integral limit theorems for additive functions

In [11], with the aid of "truncations" by measurable functions, we deduced necessary and sufficient conditions for the existence of nonsingular limit distributions for additive functions, which are in some sense "near" to being measurable (in particular, for measurable functions themselves). Measurability of an additive function $\phi(n)$ means that the sequence of distribution functions

$$\frac{1}{\pi(n)} \sum_{\substack{p \leq n \\ \varphi(p) < x}} 1$$

converges weakly to some limit distribution function as $n \rightarrow \infty$.

Theorems 8 and 9 of the preceding section show that for "truncations" one can use more than just measurable functions.

We introduce the following definition. We say that an additive function $\phi(n)$ is weakly measurable if there exists a distribution function $F(x)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{\substack{p \leq n \\ \varphi(p) < x}} \frac{\log p}{p} = F(x). \quad (5.1)$$

We note that weak measurability is, as usual, related only to the values of func-

tions at prime numbers.

The application of weakly measurable functions to the derivation of integral limit theorems for additive functions is carried out along the same lines as in [11]. Going over to characteristic functions, we get from (5.1)

$$\sum_{p \leq n} \frac{e^{i\xi\psi(p)} \log p}{p} = (\tau(\xi) + o(1)) \log n \quad (5.2)$$

for any real ξ , where

$$\tau(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} dF(x).$$

If $\psi(m)$ is an arbitrary real additive function, then according to Theorem 10 and Lemma 4

$$\begin{aligned} \frac{1}{n} \sum_{m=1}^n \exp i\xi (\psi(m_1) + \psi(m_2)) &= \frac{e^{-C\tau(\xi)}}{\Gamma(\tau(\xi)) \log n} \prod_{p \leq y} \left(1 + \sum_{r=1}^{\infty} \frac{e^{i\xi\psi(p^r)}}{p^r} \right) \\ &\quad \times \prod_{y < p \leq n} \left(1 + \sum_{r=1}^{\infty} \frac{e^{i\xi\psi(p^r)}}{p^r} \right) + o(1) \\ &= \frac{e^{-C(\tau(\xi)-1)}}{\Gamma(\tau(\xi))} \left(\frac{\log n}{\log y} \right)^{\tau(\xi)-1} \prod_{p \leq y} \left(1 + \sum_{r=1}^{\infty} \frac{e^{i\xi\psi(p^r)}}{p^r} \right) \left(1 - \frac{1}{p} \right) + o(1) \\ &= B(\xi) \left(\frac{\log n}{\log y} \right)^{\tau(\xi)-1} \prod_{p \leq y} \left(1 + \frac{e^{i\xi\psi(p)} - 1}{p} \right) + o(1), \end{aligned} \quad (5.3)$$

where

$$B(\xi) = \frac{e^{-C(\tau(\xi)-1)}}{\Gamma(\tau(\xi))} \prod_p \frac{\left(1 + \sum_{r=1}^{\infty} \frac{e^{i\xi\psi(p^r)}}{p^r} \right) \left(1 - \frac{1}{p} \right)}{1 + \frac{e^{i\xi\psi(p)} - 1}{p}}.$$

If we now apply an inequality of Turán and Kubilius ([9], Lemma 3.1), we get the following lower bound of the characteristic function (see also [11], Lemma 4).

Lemma 6. *Let $\psi(m)$ be a real additive function, $\phi(m)$ weakly measurable, $\phi(p^r) = \psi(p^r)$ for $r \geq 2$, and $C_n \rightarrow +\infty$. Then there exists $y = y(n)$ such that $y \rightarrow \infty$, $\log y = o(\log n)$, and*

$$\begin{aligned} \frac{1}{n} \sum_{m=1}^n \exp i\xi \frac{\psi(p) - K_n}{C_n} &= \prod_{p \leq y} \left(1 + \frac{e^{i\xi \frac{\psi(p)}{C_n}} - 1}{p} \right) \\ &\quad \times \exp i\xi \frac{-K_n + A_n(\psi) - A_y(\psi) - (A_n(\varphi) - A_y(\varphi))}{C_n} \\ &\quad + O \left(\frac{1}{C_n} \left(\sum_{y < p \leq n} \frac{(\psi(p) - \varphi(p))^2}{p} \right)^{1/2} \right) + o(1), \end{aligned} \quad (5.4)$$

where K_n is an arbitrary real sequence, and

$$A_x(\psi) = \sum_{p \leq x} \frac{\psi(p)}{p}.$$

From this it is evident that the limiting behavior of the sequence of distribution functions

$$F_n(x) = \frac{1}{n} N_n \left\{ \frac{\psi(m) - K_n}{C_n} < x \right\}$$

is characterized by the behavior of the normal and central sum of the independent random variables η_p ($p \leq y$) taking the value 0 with probability $1-1/p$ and the value $\psi(p)$ with probability $1/p$ in the case where

$$\lim_{n \rightarrow \infty} \frac{1}{C_n^2} \sum_{y < p \leq n} \frac{(\psi(p) - \varphi(p))^2}{p} = 0. \quad (5.5)$$

In connection with this we introduce the following definition (see also [11]). A normalization C_n is said to be ϕ -admissible for the function $\psi(m)$ if it satisfies condition (5.5).

The following three theorems are consequences of Lemma 4 and the central limit theorem of probability theory (see, for example, [15]) applied to the random variables η_p . They are proved exactly as Theorems 3–5 in [11]. For the remainder of this section $\psi(m)$ is a real additive function, $\phi(m)$ is weakly measurable, and

$$\psi^*(p) = \begin{cases} \psi(p), & \text{if } |\psi(p)| \leq 1, \\ 1, & \text{if } |\psi(p)| > 1, \end{cases}$$

$$\sum_p \frac{\psi^{*2}(p)}{p} = \infty.$$

Theorem 15. In order that there exist a centering sequence K_n and a ϕ -admissible normalization C_n such that there exists a nonsingular limit distribution

$$F(x) = \lim_{n \rightarrow \infty} \frac{1}{n} N_n \left\{ \frac{\psi(m) - K_n}{C_n} \leq x \right\}, \quad (5.6)$$

it is necessary and sufficient that:

1) there exist a nondecreasing function $L(u)$, $0 = L(-\infty) < L(+\infty) < +\infty$ such that when B_y is defined by the equation

$$\sum_{p \leq y} \frac{1}{p} \frac{\psi^2(p)}{B_y^2 + \psi^2(p)} = L(+\infty), \quad (5.7)$$

then for all $u \neq 0$ we get

$$\lim_{y \rightarrow \infty} \sum_{\substack{p \leq y \\ \psi(p) < u B_y}} \frac{1}{p} \frac{\psi^2(p)}{B_y^2 + \psi^2(p)} = L(u); \quad (5.8)$$

2) there exist a function $\gamma = \gamma(n) \rightarrow \infty$ such that $\log \gamma = o(\log n)$ and

$$\sum_{y < p \leq n} \frac{(\psi(p) - \varphi(p))^2}{p} = o(B_y^2). \quad (5.9)$$

Whenever these conditions are satisfied, the distribution (5.6) is valid with $C_n = B_{\gamma(n)}$ and

$$K_n = \sum_{\substack{p \leq y \\ |\Psi(p)| \leq C_n}} \frac{\Psi(p)}{p} + \sum_{y < p \leq n} \frac{\Psi(p) - \varphi(p)}{p} + C_n \left\{ \int_{|u| < 1} u dL(u) + \int_{|u| \geq 1} \frac{1}{u} dL(u) \right\}. \quad (5.10)$$

The logarithm of the characteristic function of the limit law is equal to

$$\int_{-\infty}^{\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2} \right) \frac{1+u^2}{u^2} dL(u). \quad (5.11)$$

Theorem 16. If for some $\gamma = \gamma(n) \rightarrow \infty$ such that $\log \gamma = o(\log n)$ the relation

$$\sum_{y < p \leq n} \frac{1}{p} \frac{\Psi^2(p)}{\Psi^2(p) + \sum_{y < q \leq n} \frac{(\Psi(q) - \varphi(q))^2}{q}} \rightarrow \infty \quad (5.12)$$

is satisfied, then condition (5.8) is necessary and sufficient for the existence of the nonsingular limit distribution (5.6).

Theorem 17. Suppose that for some $\gamma = \gamma(n) \rightarrow \infty$ such that $\log \gamma = o(\log n)$ the condition

$$\sum_{p < p \leq n} \frac{(\Psi(p) - \varphi(p))^2}{p} = o \left(\sum_{p \leq y} \frac{\Psi^2(p)}{p} \right) \quad (5.13)$$

is satisfied. In order that the nonsingular limit distribution (5.6) exist with

$$K_n = \sum_{p \leq y} \frac{\Psi(p)}{p} - \sum_{y < p \leq n} \frac{\Psi(p) - \varphi(p)}{p}, \quad C_n = \left(\sum_{p \leq y} \frac{\Psi^2(p)}{p} \right)^{1/2}, \quad (5.14)$$

it is necessary and sufficient that there exist a nondecreasing function $K(u)$ with variation 1 such that for $u \neq 0$

$$\lim_{n \rightarrow \infty} \frac{1}{C_n^2} \sum_{\substack{p \leq y \\ \Psi(p) < u C_n}} \frac{\Psi^2(p)}{p} = K(u). \quad (5.15)$$

The logarithm of the characteristic function of the limit law is equal to

$$\int_{-\infty}^{\infty} \left(e^{itu} - 1 - itu \right) \frac{dK(u)}{u^2}. \quad (5.16)$$

Thus every weakly measurable function $\phi(m)$ generates a class of real additive functions H_ϕ , to which are related functions $\psi(m)$ satisfying condition (5.15) and such that

$$\sum_p \frac{\psi^{*2}(p)}{p} = \infty.$$

As was proved in [11], there exist infinitely many disjoint classes H_ϕ .

Let E be the set of prime numbers included in the intervals (e^{n^2}, e^{n^2+1}) , $n = 1, 2, 3, \dots$. We shall show that additive functions $\psi(m)$ satisfying the condition $\frac{1}{2} \leq |\psi(p)| \leq 1$ for $p \notin E$ and $\psi(p) = \sqrt{p}$ for $p \in E$ do not belong to any of the classes H_ϕ generated by a measurable function ϕ ; on the other hand, all these functions belong to the class H_{ϕ_0} generated by a weakly measurable function $\phi_0(m)$ such that $\phi_0(p) = 0$ for $p \notin E$ and $\phi_0(p) = \sqrt{p}$ for $p \in E$.

Actually, let $\phi(m)$ be measurable, and B a constant. Then

$$\begin{aligned} & \frac{\sum_{\substack{\frac{n^2+1}{e^2} < p < e^{n^2+1}}} \frac{(\psi(p) - \varphi(p))^2}{p}}{\sum_{p < e^{n^2+1}} \frac{\psi^2(p)}{p}} > \frac{\sum_{\substack{\frac{n^2+1}{e^2} < p < e^{n^2+1} \\ \varphi(p) \leq B}} \frac{(\psi(p) - \varphi(p))^2}{p}}{\sum_{p < e^{\frac{n^2+1}{2}}} 1} \\ & > \frac{1}{4} n^2 e^{-\frac{n^2}{2}} \sum_{\substack{e^{n^2} < p < e^{n^2+1} \\ \varphi(p) \leq B}} \frac{(\psi(p) - \varphi(p))^2}{p} \\ & > \frac{1}{4} n^2 e^{-\frac{n^2}{2}} \left\{ \frac{1}{2} \sum_{\substack{e^{n^2} < p < e^{n^2+1} \\ \varphi(p) \leq B}} \frac{\psi^2(p)}{p} - B^2 \sum_{e^{n^2} < p < e^{n^2+1}} \frac{1}{p} \right\} \\ & > \frac{1}{4} n^2 e^{-\frac{n^2}{2}} \left\{ \sum_{\substack{e^{n^2} < p < e^{n^2+1} \\ \varphi(p) \leq B}} 1 + O\left(\frac{B^2}{n^2}\right) \right\}. \end{aligned} \quad (5.17)$$

If B is sufficiently large, then the last sum is greater than $0.8e^{n^2/n^2}$ by virtue of the measurability of ϕ . Therefore from (5.17) it follows that for $n > n_0$

$$\frac{\sum_{\substack{\frac{n^2+1}{e^2} < p < e^{n^2+1}}} \frac{(\psi(p) - \varphi(p))^2}{p}}{\sum_{p < e^{n^2+1}} \frac{\psi^2(p)}{p}} > 0.1e^{\frac{n^2}{2}},$$

and if $\log y(n) = o(\log n)$, then

1) Translator's note. This and the bound in the next formula are apparently typographical errors.

$$\overline{\lim} \frac{\sum_{y < p \leq n} \frac{(\psi(p) - \varphi(p))^2}{p}}{\sum_{p \leq y} \frac{\psi^2(p)}{p}} \geq \overline{\lim} \frac{\sum_{V\bar{n} < p \leq n} \frac{(\psi(p) - \varphi(p))^2}{p}}{\sum_{p \leq V\bar{n}} \frac{\psi^2(p)}{p}} = +\infty;$$

thus $\psi(m)$ does not belong to H_ϕ for any measurable function $\phi(m)$. In §3 it was shown that

$$\sum_{p \in E} \frac{1}{p} < \infty,$$

and consequently $\phi_0(m)$ is weakly measurable. That $\psi \in H_{\phi_0}$ follows from the definition of ψ and H_{ϕ_0} .

The following theorem, which is a generalization of a theorem of Erdős and Kac [12], shows, in particular, that for the corresponding normalization and centering the values of the function $\psi(m)$ are distributed asymptotically according to a normal law.

Theorem 17. *Let $\psi(m)$ be a real additive function, A a constant. If*

$$1) \quad \sum_{|\psi(p)| \leq A} \frac{\psi^2(p)}{p} = \infty, \quad (5.18)$$

$$2) \quad \sum_{|\psi(p)| > A} \frac{1}{p} < \infty, \quad (5.19)$$

then as $n \rightarrow \infty$

$$\frac{1}{n} N_n \left\{ \psi(m) - \sum_{\substack{p \leq n \\ |\psi(p)| \leq A}} \frac{\psi(p)}{p} < x \sqrt{\sum_{\substack{p \leq n \\ |\psi(p)| \leq A}} \frac{\psi^2(p)}{p}} \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du. \quad (5.20)$$

Proof. Let $\phi(m)$ be an additive function defined by the conditions

$$\varphi(p) = \begin{cases} 0, & \text{if } |\psi(p)| \leq A, \\ \psi(p), & \text{if } |\psi(p)| > A, \end{cases}$$

and $\phi(p^r) = \psi(p^r)$ for $r \geq 2$. We will show that $\phi(m)$ is weakly measurable. In fact, for $x > 0$

$$\sum_{\substack{p \leq n \\ \varphi(p) < x}} \frac{\log p}{p} = \sum_{\substack{p \leq n \\ |\psi(p)| \leq A}} \frac{\log p}{p} + \sum_{\substack{p \leq n \\ \varphi(p) \leq x \\ |\psi(p)| > A}} \frac{\log p}{p}.$$

From (5.19) it follows that

$$\sum_{\substack{p \leq n \\ |\psi(p)| > A}} \frac{\log p}{p} = o(\log n),$$

whence

$$\sum_{\substack{p \leq n \\ \Psi(p) < x}} \frac{\log p}{p} = \log n + o(\log n) \quad (x > 0). \quad (5.21)$$

Analogously, for $x < 0$

$$\sum_{\substack{p \leq n \\ \Psi(p) < x}} \frac{\log p}{p} = \sum_{\substack{p \leq n \\ |\Psi(p)| > A \\ \Psi(p) < x}} \frac{\log p}{p} = o(\log n). \quad (5.22)$$

Setting

$$K_n = \sum_{\substack{p \leq n \\ |\Psi(p)| \leq A}} \frac{\Psi(p)}{p}, \quad C_n = \left(\sum_{\substack{p \leq n \\ |\Psi(p)| \leq A}} \frac{\Psi^2(p)}{p} \right)^{1/2} \quad (5.23)$$

in Lemma 6 and computing

$$\prod_{p \leq y} \left(1 + \frac{e^{\frac{i\xi \Psi(p)}{C_n}} - 1}{p} \right) = \exp \sum_{p \leq y} \frac{e^{\frac{i\xi \Psi(p)}{C_n}} - 1}{p} + o(1), \quad (5.24)$$

we get

$$\begin{aligned} & \frac{1}{n} \sum_{m=1}^n \exp i\xi \frac{\Psi(m) - K_n}{C_n} = \exp \left\{ \sum_{p \leq y} \frac{e^{\frac{i\xi \Psi(p)}{C_n}} - 1}{p} \right. \\ & \left. - \frac{i\xi}{C_n} \sum_{\substack{p \leq n \\ |\Psi(p)| \leq A}} \frac{\Psi(p)}{p} + \frac{i\xi}{C_n} \sum_{y < p \leq n} \frac{\Psi(p)}{p} \right\} + O \left(\frac{1}{C_n} \left(\sum_{\substack{y < p \leq n \\ |\Psi(p)| \leq A}} \frac{\Psi^2(p)}{p} \right)^{1/2} \right) \\ & + o(1) = \exp \left\{ \sum_{\substack{p \leq y \\ |\Psi(p)| \leq A}} \frac{e^{\frac{i\xi \Psi(p)}{C_n}} - 1 - i\xi \frac{\Psi(p)}{C_n}}{p} \right. \\ & \left. + \sum_{\substack{p \leq y \\ |\Psi(p)| > A}} \frac{e^{\frac{i\xi \Psi(p)}{C_n}} - 1}{p} \right\} + O \left(\frac{1}{C_n} \left(\log \frac{\log n}{\log y} \right)^{\frac{1}{2}} \right) + o(1). \quad (5.25) \end{aligned}$$

Since

$$\begin{aligned} & \sum_{\substack{p \leq y \\ |\Psi(p)| \leq A}} \frac{e^{\frac{i\xi \Psi(p)}{C_n}} - 1 - i\xi \frac{\Psi(p)}{C_n}}{p} = -\frac{\xi^2}{2C_n^2} \sum_{\substack{p \leq y \\ |\Psi(p)| \leq A}} \frac{\Psi^2(p)}{p} \\ & + O \left(\frac{1}{C_n^3} \sum_{\substack{p \leq y \\ |\Psi(p)| \leq A}} \frac{\Psi^2(p)}{p} \right) = -\frac{\xi^2}{2} \frac{C_y^2}{C_n^2} + O \left(\frac{C_y^2}{C_n^2} \right), \end{aligned}$$

by choosing $y = y(n)$ so that

$$\log y = o(\log n) \quad \text{and} \quad \frac{1}{C_n} \left(\log \frac{\log n}{\log y} \right)^{1/2} \rightarrow 0,$$

we find that $C_y/C_n \rightarrow 1$ and

$$\frac{1}{n} \sum_{m \leq n} \exp i \xi \frac{\psi(m) - K_n}{C_n} \rightarrow e^{-\frac{\xi^2}{2}},$$

whence follows the assertion being proved.

§6. On the measurable distribution of fractional parts of additive functions

The theorems of §4 which give sufficient conditions under which the limiting relation

$$\sum_{n \leq x} f(n) = o(x),$$

where $f(n)$ is a multiplicative function, can be applied, with the aid of Weyl's criterion, to prove the uniformity of the distribution of fractional parts of additive functions.

First we note that the following general assertion was established in the proof of Theorem 12: if E is a set of prime numbers of positive weak asymptotic density, i.e.

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{\substack{p \leq N \\ p \in E}} \frac{\log p}{p} = \delta > 0, \quad (6.1)$$

and $f(n)$ is a multiplicative function for which $|f(n)| \leq 1$ and

$$\sum_{\substack{n \leq N \\ n \in (E)}} f(n) = o\left(\sum_{\substack{n \leq N \\ n \in (E)}} 1\right), \quad (6.2)$$

then

$$\sum_{n \leq N} f(n) = o(N). \quad (6.3)$$

The following is an obvious corollary of this fact and of Weyl's criterion.

Theorem 18. *If the fractional parts of a real additive function $\psi(m)$ are uniformly distributed as m runs over the set (E) , i.e. for $0 \leq \alpha < \beta \leq 1$*

$$\sum_{\substack{n \leq N \\ n \in (E) \\ \alpha < \{\psi(n)\} < \beta}} 1 \sim (\beta - \alpha) \sum_{\substack{n \leq N \\ n \in (E)}} 1, \quad (6.4)$$

then the sequence of fractional parts $\{\psi(m)\}$, $m = 1, 2, 3, \dots$, is uniformly distributed modulo 1.

Actually, it follows from (6.4) that for any integer $k \neq 0$

$$\sum_{\substack{n \leq N \\ n \in E}} e^{2\pi i k \psi(n)} = o\left(\sum_{\substack{n \leq N \\ n \in E}} 1\right);$$

hence on the basis of the preceding remark we have

$$\sum_{n \leq N} e^{2\pi i k \psi(n)} = o(N).$$

This result shows that in many cases to establish the uniform distribution of the fractional parts of an additive function it is sufficient to have information only about the behavior of the function on some set of primes satisfying condition (6.1).

We shall say that a real additive function is measurable modulo 1 on a set E if there exists a distribution function $F(x)$ at each of whose points of continuity

$$\sum_{\substack{p \leq N \\ p \in E \\ \{\psi(p)\} < x}} 1 \sim F(x) \sum_{\substack{p \leq N \\ p \in E}} 1. \quad (6.5)$$

For all that follows E is a set of primes satisfying condition (6.1), and $\psi(m)$ is measurable modulo 1 on E .

Theorem 19. *If for any integer $k \neq 0$*

$$\sum_{p \in E} \frac{\|k\psi(p)\|^2}{p} = \infty, \quad (6.6)$$

then the fractional parts of $\psi(m)$ are uniformly distributed modulo 1.

Proof. We consider the multiplicative function $e^{2\pi i k \psi(n)}$. From (6.5) it follows that

$$\sum_{\substack{p \leq N \\ p \in E}} e^{2\pi i k \psi(p)} \sim \tau(2\pi k) \sum_{\substack{p \leq N \\ p \in E}} 1 \sim \tau(2\pi k) \delta \frac{N}{\log N},$$

where $\tau(\xi)$ is the characteristic function of the distribution $F(x)$. Moreover,

$$\begin{aligned} \sum_{p \in E} \frac{1 - \operatorname{Re} e^{2\pi i k \psi(p)}}{p} &= \sum_{p \in E} \frac{1 - \cos 2\pi k \psi(p)}{p} \\ &= 2 \sum_{p \in E} \frac{\sin^2 \pi \|k\psi(p)\|}{p} \geq 8 \sum_{p \in E} \frac{\|k\psi(p)\|^2}{p} = \infty, \end{aligned}$$

so that $e^{2\pi i k \psi(n)}$ satisfies all the conditions of Theorem 10. Therefore

$$\sum_{n \leq N} e^{2\pi i k \psi(n)} = o(N)$$

and the application of Weyl's criterion completes the proof.

Corollary 1. *If the distribution $F(x)$ of (6.5) is not an arithmetic distribution with*

a rational step, then the fractional parts of $\psi(m)$ are uniformly distributed modulo 1.

Actually, in this case for any integer $k \neq 0$

$$\operatorname{Re} \tau(2\pi k) = \int_0^1 \cos 2\pi kx \, dF(x) < 1$$

(see also [11]).

Corollary 2. If

$$\lim_{\substack{p \rightarrow \infty \\ p \in E}} \psi(p) = 0 \quad (6.7)$$

and

$$\sum_{p \in E} \frac{\psi^2(p)}{p} = \infty, \quad (6.8)$$

then the fractional parts of $\psi(m)$ are uniformly distributed modulo 1.

This is a generalization of a well-known theorem of Erdős [13] which can be obtained from it by taking E to be the set of all prime numbers.

Actually (6.5) follows from (1.11) with

$$F(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Moreover, the divergence of all series of the form (6.6) follows from (6.7) and (6.8).

The conditions of Theorem 18 are obviously met if $\psi(p) = \theta$ for $p \in E$, where θ is an irrational constant. For the case where E is a set of primes from some arithmetic progression, it is possible to get an estimate of the error, i.e. the remainder

$$\sum_{\substack{n \leq N \\ \alpha < \{\psi(n)\} \leq \beta}} 1 - (\beta - \alpha) N.$$

Applying Theorem 11 to the multiplicative function $f(n) = e^{2\pi i k \psi(n)}$, we get

$$\sum_{n \leq N} e^{2\pi i k \psi(n)} \ll N (\log N)^{\frac{\max(\cos 2\pi k \theta, 0) - 1}{\varphi(D)}}, \quad (6.9)$$

where D is the difference of the progression. Since

$$\cos 2\pi k \theta - 1 = 2 \sin^2 \pi k \theta \geq 8 \|k\theta\|^2,$$

it follows from (1.13) that

$$\sum_{n \leq N} e^{2\pi i k \psi(n)} \ll N (\log N)^{-\frac{8}{\varphi(D)} \min\left(\|k\theta\|^2, \frac{1}{8}\right)}. \quad (6.10)$$

This last inequality shows that the estimate of the error depends on the character of the rational approximations of the number θ . If, for example, θ is an algebraic number, then by a theorem of Roth [16] we have

$$\|k\theta\| > \frac{1}{k^{1+\varepsilon}} \quad \text{for } k > k_0(\varepsilon). \quad (6.11)$$

This implies

Theorem 20. Let θ be an algebraic number of degree greater than 1, $(D, l) = 1$, and $\psi(m)$ a real additive function such that $\psi(p) = \theta$ if $p \equiv l \pmod{D}$. Then for $0 \leq \alpha < \beta \leq 1$

$$\frac{1}{N} \sum_{\substack{n \leq N \\ \alpha < \{\psi(n)\} \leq \beta}} 1 = \beta - \alpha + O\left(\frac{1}{(\log \log N)^{\frac{1}{2} - \varepsilon}}\right). \quad (6.12)$$

For the proof we make use of a well-known inequality of Erdős and Turán [17]:

$$\left| \frac{1}{N} \sum_{\substack{n \leq N \\ \alpha < \{a_n\} \leq \beta}} 1 - (\beta - \alpha) \right| < C \left(\frac{1}{T} + \frac{1}{N} \sum_{m \leq T} \frac{1}{m} \left| \sum_{n \leq N} e^{2\pi i m a_n} \right| \right),$$

where C is an absolute constant, and $T > 0$ is arbitrary.

Setting $a_n = \psi(n)$ and taking advantage of (6.10) and (6.11), we get

$$\begin{aligned} \left| \frac{1}{N} \sum_{\substack{n \leq N \\ \alpha < \{\psi(n)\} \leq \beta}} 1 - (\beta - \alpha) \right| &< C \left\{ \frac{1}{T} + \sum_{m \leq T} \frac{1}{m} (\log N)^{-\frac{8}{\varphi(D)} \min\left(\|m\theta\|^2, \frac{1}{8}\right)} \right\} \\ &\ll \frac{1}{T} + \sum_{m \leq T} \frac{1}{m} (\log N)^{-\frac{8}{\varphi(D)} \min\left(\frac{1}{m^{2+2\varepsilon}}, \frac{1}{8}\right)} \\ &\ll \frac{1}{T} + \log T \cdot \exp \left\{ -\frac{8 \log \log N}{\varphi(D) T^{2+2\varepsilon}} \right\}. \end{aligned}$$

The proof is completed by choosing $T = (\log \log N)^{\frac{1}{2} - \varepsilon}$.

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